

# CHERN CLASSES OF PROALGEBRAIC VARIETIES AND MOTIVIC MEASURES

SHOJI YOKURA\*

ABSTRACT. Michael Gromov has recently initiated what he calls “symbolic algebraic geometry”, in which objects are proalgebraic varieties: a proalgebraic variety is by definition the projective limit of a projective system of algebraic varieties. In this paper we construct Chern–Schwartz–MacPherson classes of proalgebraic varieties, by introducing the notion of “proconstructible functions” and “ $\chi$ -stable proconstructible functions” and using the Fulton–MacPherson’s Bivariant Theory. As a “motivic” version of a  $\chi$ -stable proconstructible function,  $\Gamma$ -stable constructible functions are introduced. This construction naturally generalizes the so-called motivic measure and motivic integration. For the Nash arc space  $\mathcal{L}(X)$  of an algebraic variety  $X$ , the proconstructible set is equivalent to the so-called cylinder set or constructible set in the arc space.

## §1 INTRODUCTION

This work is motivated by Gromov’s papers [Grom 1, Grom 2] and also by [Y4].

A pro-category was first introduced by A. Grothendieck [Grot] and it was used to develop the Etale Homotopy Theory [AM] and Shape Theory (e.g., see [Bor], [Ed], [MS], etc.) and so on. A pro-algebraic variety is defined to be a projective system of complex algebraic varieties and a proalgebraic variety is defined to be the projective limit of a pro-algebraic variety. In [Grom 1] M. Gromov investigated the *surjunctivity* [Got], i.e., *being either surjective or non-injective*, in the category of proalgebraic varieties. The original or classical surjunctivity theorem is the so-called *Ax’ Theorem*, saying that every regular selfmapping of a complex algebraic variety is surjunctive; thus if it is injective then it has to be surjective (cf, [Ax], [BBR], [Bo], [Kurd], [New], [Par], etc.). In [Grom 1] he initiated what he calls “symbolic algebraic geometry” and in its Abstract he says “... The paper intends to bring out relations between model theory, algebraic geometry, and symbolic dynamics.”

Our interest at the moment is not a further investigation concerning Ax-type theorems, but characteristic classes, in particular, Chern classes of proalgebraic varieties. One motivation of this is that in [Grom 1] he uses the term of ‘proconstructible set or space’ at

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several places; so we ask ourselves : “whatever the definition of ‘proconstructible set’ is, what would be the Chern–Schwartz–MacPherson class of a proalgebraic variety ?” This way of thinking is simply motivated by the fact that ‘constructible set’ is nothing but ‘constructible function’ and that in turn it is nothing but ‘homology class’ via the Chern–Schwartz–MacPherson class transformation  $c_* : F \rightarrow H_*$  (see [M], [BS], [Sc1, Sc2]). This transformation is the unique natural transformation from the covariant functor  $F$  of constructible functions to the covariant homology functor  $H_*$ , satisfying the normalization condition that for a smooth variety  $X$  the value  $c_*(\mathbb{1}_X)$  of the characteristic function  $\mathbb{1}_X$  is the Poincaré dual of the total Chern cohomology class  $c^*(X)$ . The unique existence of such a transformation was conjectured by Pierre Deligne and Alexander Grothendieck, and finally was solved affirmatively by Robert MacPherson [M]. Later it was shown by Jean-Paul Brasselet and Marie-Hélène Schwartz [BS] that it is isomorphic to the Schwartz class [Sc1, Sc2] via the Alexander duality isomorphism. This is the reason for naming “Chern–Schwartz–MacPherson”.

Another motivation is that in [Y4] we investigated some “quasi-bivariant” Chern classes, using the projective system of resolutions of singularities and its limits. (For bivariant theories, see [FM], [Br], [BSY], [E2], [EY1, 2], [F1], [G1], [Sa2], [Sch2], [Y1, 2, 3], [Z1, 2], etc.). What we did in [Y4] can be put in as follows: we make some bivariant classes obtained by all the different resolutions of singularities “equivalent” or “the same” in some inductive limit.

A very simple example of a proalgebraic variety is the Cartesian product  $X^{\mathbb{N}}$  of an infinite countable copies of a complex algebraic variety  $X$ , which is one of the main objects treated in [Grom 1], i.e., a proalgebraic variety associated with a symbolic dynamic. Then, what would be the Chern–Schwartz–MacPherson of  $X^{\mathbb{N}}$  ? In particular, what would be the “*Euler–Poincaré (pro)characteristic*” of  $X^{\mathbb{N}}$  ? Our answers are that they are respectively  $c_*(X)$  and  $\chi(X)$  in some sense, which will be clarified later. It is this very simple observation (which looked stupid, nonsensical or meaningless at the beginning) that led us to the present work, which naturally led us to motivic measures, which have been actively studied by many people (e.g., see [Cr], [DL 1], [DL 2], [Kon], [Loo], [Ve] etc.).

Taking the degree of the 0-dimensional component or taking integration of the Chern–Schwartz–MacPherson class homomorphism  $c_* : F(X) \rightarrow H_*(X)$  gives us the Euler Poincaré characteristic (homomorphism)

$$(1.1) \quad \chi : F(X) \rightarrow \mathbb{Z} \quad \text{described by} \quad \chi(\alpha) = \sum_{n \in \mathbb{Z}} n \chi(\alpha^{-1}(n)).$$

Another notation for this, putting emphasis on integration, is

$$(1.1.a) \quad \int_X \alpha d\chi = \sum_{n \in \mathbb{Z}} n \chi(\alpha^{-1}(n)).$$

Also, for a function  $f(\alpha)$ , we can define the following

$$(1.1.b) \quad \int_X f(\alpha) d\chi = \sum_{n \in \mathbb{Z}} f(n) \chi(\alpha^{-1}(n)).$$

Which is a generalized version of (1.1.a).

Since a constructible function  $\alpha$  on  $X$  can be expressed as a finite linear combination of characteristic functions  $\mathbb{1}_W$  of subvarieties  $W$  of  $X$ ,  $\alpha = \sum n_W \mathbb{1}_W$ , (1.1) is simply expressed as  $\chi(\sum n_W \mathbb{1}_W) = \sum n_W \chi(W)$ . Another distinguished component of the  $c_*$  is the “top”-dimensional component. For  $\alpha = \sum n_W \mathbb{1}_W$  we have  $c_*(\alpha) = \sum n_W c_*(W)$ . Since the top-dimensional component of each  $c_*(W)$  is the fundamental class  $[W]$ , the “top”-dimensional part or the “fundamental class” part of  $c_*(\alpha)$  shall be denoted by  $[c_*(\alpha)]$  and  $[c_*(\alpha)] = \sum n_W [W]$ . So, if we “regard” the fundamental class  $[W]$  as a class  $[W]$  of the variety  $W$  in the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$  of complex algebraic varieties, then we can “get” the homomorphism  $\Gamma : F(X) \xrightarrow{c_*} H_*(X) \xrightarrow{[\ ]} K_0(\mathcal{V}_{\mathbb{C}})$ . This way of thinking is, however, not quite right, because the homomorphism  $[\ ] : H_*(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})$  is not well-defined. But, the homomorphism

$$(1.2) \quad \Gamma : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}}) \quad \text{defined by} \quad \Gamma(\alpha) = \sum_{n \in \mathbb{Z}} n [\alpha^{-1}(n)]$$

is a well-defined homomorphism, which is called the *Grothendieck class homomorphism*. Simply expressing, it is  $\Gamma(\sum n_W \mathbb{1}_W) = \sum n_W [W]$ . Thus (1.2) is a “motivic” version of (1.1). Or as a “motivic” version of (1.1.a), (1.2) can be expressed by  $\int_X \alpha d\Gamma = \sum_{n \in \mathbb{Z}} n [\alpha^{-1}(n)]$ . And like (1.1.b), for a function  $f(\alpha)$  we have

$$\int_X f(\alpha) d\Gamma = \sum_{n \in \mathbb{Z}} f(n) [\alpha^{-1}(n)].$$

Mimicking or abusing the above notations, we can also consider  $c_*(\alpha) = \int_X \alpha dc_* = \sum_{n \in \mathbb{Z}} n c_* (\alpha^{-1}(n))$ . If we let  $c_i : F \rightarrow H_{2i}$  be the  $2i$ -dimensional component of  $c_* : F \rightarrow H_*$ ,

$$\text{we can also have } c_i(\alpha) = \int_X \alpha dc_i = \sum_{n \in \mathbb{Z}} n c_i (\alpha^{-1}(n)).$$

What we do in this paper is to extend or generalize characteristic classes, in particular, the Chern–Schwartz–MacPherson class  $c_* : F(X) \rightarrow H_*(X)$ , of complex (possibly singular) algebraic varieties to the category of proalgebraic varieties. The first thing that we have to consider is how to define the proalgebraic version of  $F(X)$ , namely, how to define a reasonable notion of “*proconstructible function*”, i.e., a proalgebraic version of a constructible function.

Especially, one can extend the above two homomorphisms  $\chi : F(X) \rightarrow \mathbb{Z}$  and  $\Gamma : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})$  to the category of proalgebraic varieties and it turns out that the extended version include the so-called motivic measures as special cases. The key for extending these is that they are both multiplicative, while the other individual components of  $c_*$  are not multiplicative and thus one cannot extend those to the category of proalgebraic varieties.

The organization of the paper is as follows. In §2 we just consider Chern class of a projective system of algebraic varieties, which is very straightforward. In §3 we define

“*proconstructible functions*” and we formulate Chern–Schwartz–MacPherson classes of some proalgebraic varieties, making a full use of the Bivariant Theory introduced by William Fulton and Robert MacPherson [FM], in particular a bivariant Chern class [Br]. The key of the results obtained in §3 is the constancy of the Euler–Poincaré characteristics of the fibers of each structure morphism in a projective system of varieties. If this constancy is not satisfied, we need a stronger requirement on the proconstructible function, and thus in §4 we introduce the notion of  $\chi$ -stable proconstructible functions. In §5 we discuss motivic measures and furthermore, hinted by the definition of  $\chi$ -stable proconstructible functions, we introduce the notion of  $\Gamma$ -stable (or “*motivic*” stable) proconstructible functions, and thus we get a generalization of so-called *stable constructible functions*. In §6 we discuss the proalgebraic version of integration with respect to the proalgebraic Euler–Poincaré characteristic  $\chi^{\text{pro}}$  and the proalgebraic Grothendieck “*motivic*” class homomorphism  $\Gamma^{\text{pro}}$ . In §7 we introduce the notion of *proresolution of singularities* and we pose a problem concerning a relationship between the Nash arc space  $\mathcal{L}(X)$  and the proresolution of  $X$ , which could be related to the problem which Nash considered in his paper [Na], in relation with Hironaka’s resolution of singularities [Hi].

Finally we point out that a natural question coming out of this way of thinking is whether one can obtain a theory of Chern–Schwartz–MacPherson classes with values in the Grothendieck ring, i.e., a “*motivic*” version of the Chern–Schwartz–MacPherson class  $c_*$ , and furthermore, if we consider the category of proalgebraic varieties, a natural question is whether one can obtain a theory of Chern–Schwartz–MacPherson classes with values in localizations of the Grothendieck ring. These remain to be seen. And for another connection of Chern–Schwartz–MacPherson class with motivic measure or integration, see Paolo Aluffi’s recent article [A].

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## §2 CHERN CLASSES OF PRO-ALGEBRAIC VARIETIES

Let  $I$  be a directed set and let  $\mathcal{C}$  be a given category. Then a projective system is, by definition, a system  $\{X_i, \pi_{ii'} : X_{i'} \rightarrow X_i (i < i'), I\}$  consisting of objects  $X_i \in \text{Obj}(\mathcal{C})$ , morphisms  $\pi_{ii'} : X_{i'} \rightarrow X_i \in \text{Mor}(\mathcal{C})$  for each  $i < i'$  and the index set  $I$ . The object  $X_i$  is called a *term* and the morphism  $\pi_{ii'} : X_{i'} \rightarrow X_i$  a *bonding morphism* or *structure morphism* ([MS]). The projective system  $\{X_i, \pi_{ii'} : X_{i'} \rightarrow X_i (i < i'), I\}$  is sometimes simply denoted by  $\{X_i\}_{i \in I}$ .

Given a category  $\mathcal{C}$ ,  $\text{Pro-}\mathcal{C}$  is the category whose objects are projective systems  $X = \{X_i\}_{i \in I}$  in  $\mathcal{C}$  and whose set of morphisms from  $X = \{X_i\}_{i \in I}$  to  $Y = \{Y_j\}_{j \in J}$  is

$$\text{Pro-}\mathcal{C}(X, Y) := \varprojlim_J (\varinjlim_I \mathcal{C}(X_i, Y_j)).$$

This definition is not crystal clear, but a more down-to-earth definition is the following (e.g., see [Fox] or [MS]): A morphism  $f : X \rightarrow Y$  consists of a map  $\theta : J \rightarrow I$  (not necessarily order preserving) and morphisms  $f_j : X_{\theta(j)} \rightarrow Y_j$  for each  $j \in J$ , subject to the condition that if  $j < j'$  in  $J$  then for some  $i \in I$  such that  $i > \theta(j)$  and  $i > \theta(j')$ , the following diagram commutes

$$\begin{array}{ccc}
& X_i & \\
\pi_{\theta(j')i} \swarrow & & \searrow \pi_{\theta(j)i} \\
X_{\theta(j')} & & X_{\theta(j)} \\
f_{j'} \downarrow & & \downarrow f_j \\
Y_{j'} & \xrightarrow{\rho_{jj'}} & Y_j
\end{array}$$

Given a projective system  $X = \{X_i\}_{i \in I} \in \text{Pro-}\mathcal{C}$ , the projective limit  $X_\infty := \varprojlim X_i$  may not belong to the source category  $\mathcal{C}$ . For a certain sufficient condition for the existence of the projective limit in the category  $\mathcal{C}$ , see [MS] for example.

An object in  $\text{Pro-}\mathcal{C}$  is called a *pro-object*. A projective system of algebraic varieties is called a *pro-algebraic variety* and its projective limit is called a *proalgebraic variety*, which may not be an algebraic variety but simply a topological space.

Let  $T : \mathcal{C} \rightarrow \mathcal{D}$  be a covariant functor between two categories  $\mathcal{C}, \mathcal{D}$ . Obviously the covariant functor  $T$  extends to a covariant pro-functor

$$\text{Pro-}T : \text{Pro-}\mathcal{C} \rightarrow \text{Pro-}\mathcal{D}$$

defined by  $\text{Pro-}T(\{X_i\}_{i \in I}) := \{T(X_i)\}_{i \in I}$ . Let  $T_1, T_2 : \mathcal{C} \rightarrow \mathcal{D}$  be two covariant functors and  $N : T_1 \rightarrow T_2$  be a natural transformation between the two functors  $T_1$  and  $T_2$ . Then the natural transformation  $N : T_1 \rightarrow T_2$  extends to a natural pro-transformation

$$\text{Pro-}N : \text{Pro-}T_1 \rightarrow \text{Pro-}T_2.$$

Thus a pro-algebraic version of the Chern–Schwartz–MacPherson class is straightforward, i.e., we have

$$\text{Pro-}c_* : \text{Pro-}F \rightarrow \text{Pro-}H_*$$

and thus there is nothing to be done for it. In this case, the characteristic pro-function  $\mathbb{1}_X$  of the pro-algebraic variety  $X = \{X_i\}_{i \in I}$  should be simply  $\mathbb{1}_X := \{\mathbb{1}_{X_i}\}_{i \in I}$  and thus the pro-version of the Chern–Schwartz–MacPherson class of the pro-algebraic variety  $X = \{X_i\}_{i \in I}$  is simply  $\text{Pro-}c_*(X) = \{c_*(X_i)\}_{i \in I}$ .

What we want to do is its proalgebraic version.

*Remark (2.1).* In Etale Homotopy Theory [AM] and Shape Theory (e.g., see [Bor], [Ed], [MS]) they stay in the pro-category and do not consider limits and colimits, because doing so throw away some geometric informations.

### §3 PROCONSTRUCTIBLE FUNCTIONS AND CHERN CLASSES OF PROALGEBRAIC VARIETIES

As mentioned above, a pro-morphism between two pro-objects is quite complicated. However, it follows from [MS] that the pro-morphism can be described more naturally as a so-called *level preserving pro-morphism*. Suppose that we have two pro-algebraic varieties  $X = \{X_\gamma\}_{\gamma \in \Gamma}$  and  $Y = \{Y_\lambda\}_{\lambda \in \Lambda}$ . Then a pro-algebraic morphism  $\Phi = \{f_\lambda\}_{\lambda \in \Lambda} : X \rightarrow Y$

is described as follows: there is an order-preserving map  $\xi : \Lambda \rightarrow \Gamma$ , i.e.,  $\xi(\lambda) < \xi(\mu)$  for  $\lambda < \mu$ , and for each  $\lambda \in \Lambda$  there is a morphism  $f_\lambda : X_{\xi(\lambda)} \rightarrow Y_\lambda$  such that for  $\lambda < \mu$  the following diagram commutes:

$$\begin{array}{ccc} X_{\xi(\mu)} & \xrightarrow{f_\mu} & Y_\mu \\ \pi_{\xi(\lambda)\xi(\mu)} \downarrow & & \downarrow \rho_{\lambda\mu} \\ X_{\xi(\lambda)} & \xrightarrow{f_\lambda} & Y_\lambda, \end{array}$$

Then, the projective limit of the system  $\{f_\lambda\}$  is a morphism from the proalgebraic variety  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$  to the proalgebraic variety  $Y_\infty = \varprojlim_{\gamma \in \Gamma} Y_\gamma$ . It is called a proalgebraic morphism and denoted by  $f_\infty : X_\infty \rightarrow Y_\infty$ .

The projective system  $\{X_\gamma, \pi_{\gamma\delta}(\gamma < \delta)\}$  induces the projective system of abelian groups of constructible functions  $\{F(X_\gamma), (\pi_{\gamma\delta})_* : F(X_\delta) \rightarrow F(X_\gamma)(\gamma < \delta)\}$ . And a system of morphisms  $f_\lambda : X_{\xi(\lambda)} \rightarrow Y_\lambda$  induces the system of homomorphisms  $f_{\lambda*} : F(X_{\xi(\lambda)}) \rightarrow F(Y_\lambda)$ . Thus the system of commutative diagrams

$$\begin{array}{ccc} F(X_{\xi(\mu)}) & \xrightarrow{f_{\mu*}} & F(Y_\mu) \\ \pi_{\xi(\lambda)\xi(\mu)*} \downarrow & & \downarrow \rho_{\lambda\mu*} \\ F(X_{\xi(\lambda)}) & \xrightarrow{f_{\lambda*}} & F(Y_\lambda), \end{array}$$

induces the homomorphism  $f_{*\infty} : \varprojlim_{\gamma \in \Gamma} F(X_\gamma) \rightarrow \varprojlim_{\lambda \in \Lambda} F(Y_\lambda)$ . Similarly we get the homomorphism of the projective limits of homology groups  $f_{*\infty} : \varprojlim_{\gamma \in \Gamma} H_*(X_\gamma) \rightarrow \varprojlim_{\lambda \in \Lambda} H_*(Y_\lambda)$ . Furthermore the commutative diagram of Chern–Schwartz–MacPherson class homomorphisms

$$\begin{array}{ccc} F(X_\mu) & \xrightarrow{c_*} & H_*(X_\mu) \\ \pi_{\lambda\mu*} \downarrow & & \downarrow \pi_{\lambda\mu*} \\ F(X_\lambda) & \xrightarrow{c_*} & H_*(X_\lambda), \end{array}$$

induces the projective limit of Chern–Schwartz–MacPherson classes:

$$c_{*\infty} : \varprojlim_{\lambda \in \Lambda} F(X_\lambda) \rightarrow \varprojlim_{\lambda \in \Lambda} H_*(X_\lambda)$$

So, we define, for the proalgebraic variety  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$ ,

$$\text{pro } F(X_\infty) := \varprojlim_{\lambda \in \Lambda} F(X_\lambda) \quad \text{and} \quad \text{pro } H_*(X_\infty) := \varprojlim_{\lambda \in \Lambda} H_*(X_\lambda).$$

If we define  $\text{pro } c_* : \text{pro } F \rightarrow \text{pro } H_*$  to be the above  $c_{*\infty}$  and define  $f_{\infty*}$  to be the above  $f_{*\infty}$ , then we have a naïve proalgebraic version of the Chern–Schwartz–MacPherson class

$$\mathrm{pro} \, c_* : \mathrm{pro} \, F \rightarrow \mathrm{pro} \, H_*,$$

i.e., for a proalgebraic morphism  $f_\infty : X_\infty \rightarrow Y_\infty$  we have the commutative diagram

$$\begin{array}{ccc} \mathrm{pro} \, F(X_\infty) & \xrightarrow{\mathrm{pro} \, c_*} & \mathrm{pro} \, H_*(X_\infty) \\ f_{\infty*} \downarrow & & \downarrow f_{\infty*} \\ \mathrm{pro} \, F(Y_\infty) & \xrightarrow{\mathrm{pro} \, c_*} & \mathrm{pro} \, H_*(Y_\infty). \end{array}$$

Note that if the directed set  $\Lambda$  is finite, then it is clear that there is an element  $0 \in \Lambda$  which is cofinal. In this case, the proalgebraic variety  $X_\infty$  is isomorphic to the variety  $X_0$  and thus the above naïve proalgebraic Chern–Schwartz–MacPherson class  $\mathrm{pro} \, c_* : \mathrm{pro} \, F \rightarrow \mathrm{pro} \, H_*$  is nothing but the original one. Thus we are of course interested in the case where the directed set  $\Lambda$  is infinite. For example, let the directed set be the natural numbers  $\mathbb{N}$  and the projective system is the sequence of complex algebraic varieties:

$$\cdots \rightarrow X_i \rightarrow \cdots \rightarrow X_3 \rightarrow X_2 \rightarrow X_1.$$

For instance, for an algebraic variety  $X$  of finite dimension, let us denote the Cartesian product  $X \times X \times \cdots \times X$  of  $n$  copies of  $X$  by  $X^n$ . From now on, the superscript  $n$  of  $X^n$  does not mean the dimension of the variety  $X$ , unless stated otherwise. And we let  $\pi_{n-1,n} : X^n \rightarrow X^{n-1}$  be the canonical projection defined by  $\pi_{n-1,n}(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_{n-1})$  and consider the infinite sequence

$$\cdots \rightarrow X^n \rightarrow X^{n-1} \rightarrow \cdots \rightarrow X^3 \rightarrow X^2 \rightarrow X.$$

Its projective limit is the infinite product  $X^\mathbb{N} = X^\infty$ , which is an important object in Gromov’s paper [Grom 1]. The progroup  $\mathrm{pro} \, F(X^\mathbb{N}) = \varprojlim_n F(X^n)$  is, by definition

$$\mathrm{pro} \, F(X^\mathbb{N}) = \left\{ (\alpha_j) \in \prod_{j=1}^{\infty} F(X^j) \mid \forall j \quad \alpha_j = (\pi_{j,j+1})_* \alpha_{j+1} \right\}.$$

The  $j$ -th component  $\alpha \in F(X^j)$  of  $(\alpha_j) \in \prod_{j=1}^{\infty} F(X^j)$  automatically determines the lower components  $\alpha_1, \alpha_2, \dots, \alpha_{j-1}$ , i.e., for each  $k < j$

$$\alpha_k = (\pi_{k,k+1})_* \cdots (\pi_{j-2,j-1})_* (\pi_{j-1,j})_* \alpha.$$

However, as to the upper components  $\alpha_k (k > j)$ , there are infinitely many choices and there is no formula to determine it. Thus, the structure of the progroup  $\mathrm{pro} \, F(X^\infty)$  is not so obvious.

Next, how do we capture an element of  $\varprojlim_{\lambda \in \Lambda} F(X_\lambda)$  as a function on the proalgebraic variety  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$ , namely, what is the value of an element  $\alpha_\infty = (\alpha_\lambda) \in \varprojlim_{\lambda \in \Lambda} F(X_\lambda) \subset \prod F(X_\lambda)$  at a point  $(x_\lambda) \in \varprojlim_{\lambda \in \Lambda} X_\lambda \subset \prod X_\lambda$ ? A very naïve definition could be

$$\alpha_\infty((x_\lambda)) = \alpha_\lambda(x_\lambda),$$

which is, however, not well-defined. Indeed, by the definition of the projective limit we have that  $\pi_{\lambda\mu}(x_\mu) = x_\lambda$  and that  $\pi_{\lambda\mu*}(\alpha_\mu) = \alpha_\lambda$  for  $\lambda < \mu$ . The equality  $\pi_{\lambda\mu*}(\alpha_\mu) = \alpha_\lambda$  implies that

$$\alpha_\lambda(x_\lambda) = \pi_{\lambda\mu*}(\alpha_\mu)(x_\lambda) = \chi\left(\pi_{\lambda\mu}^{-1}(x_\lambda); \alpha_\mu\right)$$

which is the weighted Euler–Poincaré characteristic of the fiber  $\pi_{\lambda\mu}^{-1}(x_\lambda)$ , or the sum of the 0-dimensional component of the Chern–Schwartz–MacPherson class  $c_*(\alpha_\mu|_{\pi_{\lambda\mu}^{-1}(x_\lambda)})$  of the constructible function  $\alpha_\mu|_{\pi_{\lambda\mu}^{-1}(x_\lambda)}$  on the fiber  $\pi_{\lambda\mu}^{-1}(x_\lambda)$ . Here,  $\chi(A; \alpha) := \sum_{n \in \mathbb{Z}} n\chi(A \cap \alpha^{-1}(n))$ . Thus, in general, these two equalities  $\pi_{\lambda\mu}(x_\mu) = x_\lambda$  and  $\pi_{\lambda\mu*}(\alpha_\mu) = \alpha_\lambda$  do not imply that  $\alpha_\lambda(x_\lambda) = \alpha_\mu(x_\mu)$ . Thus  $\alpha_\infty((x_\lambda)) = \alpha_\lambda(x_\lambda)$  is not well-defined. Hence, an element of the progroup  $\text{pro } F(X_\infty)$  defined above would not be a good candidate to be considered as a function on the proalgebraic variety  $X_\infty$ .

However, the equalities  $\pi_{\lambda\mu}(x_\mu) = x_\lambda$  and  $\pi_{\lambda\mu}^*(\alpha_\lambda) = \alpha_\mu$  imply that  $\alpha_\infty((x_\lambda)) = \alpha_\lambda(x_\lambda)$  is well-defined, since we have that

$$\alpha_\mu(x_\mu) = (\pi_{\lambda\mu}^*(\alpha_\lambda))(x_\mu) = \alpha_\lambda(\pi_{\lambda\mu}(x_\mu)) = \alpha_\lambda(x_\lambda).$$

So, it is reasonable to define the following

**Definition (3.1).** For a proalgebraic variety  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$ , the inductive limit of the inductive system  $\left\{F(X_\lambda), \pi_{\lambda\mu}^* : F(X_\lambda) \rightarrow F(X_\mu) (\lambda < \mu)\right\}$  is denoted by  $F^{\text{pro}}(X_\infty)$ , i.e.,

$$F^{\text{pro}}(X_\infty) := \varinjlim_{\lambda \in \Lambda} F(X_\lambda) = \bigcup_{\mu} \rho^\mu(F(X_\mu))$$

where  $\rho^\mu : F(X_\mu) \rightarrow \varinjlim_{\lambda \in \Lambda} F(X_\lambda)$  is the homomorphism sending  $\alpha_\mu$  to its equivalence class  $[\alpha_\mu]$  of  $\alpha_\mu$ . An element of the group  $F^{\text{pro}}(X_\infty)$  is called a *proconstructible* function on the proalgebraic variety  $X_\infty$ .

The proconstructible function  $[\mathbb{1}_{X_\lambda}]$  for any  $\lambda \in \Lambda$  shall be called *the procharacteristic function* on  $X_\infty$  and denoted by  $\mathbb{1}_{X_\infty}$ .

The terminology *proconstructible* is used in [Grom 1], but its definition does not seem to be given explicitly in his paper.

Definition (3.1) can be used for any contravariant functor on the category of objects. Namely, if  $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{C}'$  is a contravariant functor and  $\{X_\lambda, \pi_{\lambda\mu} : X_\mu \rightarrow X_\lambda (\lambda < \mu)\}$  is a projective system in  $\mathcal{C}$ , then for the projective limit  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$ , which itself *may not belong to* the category  $\mathcal{C}$ , we can define

$$\mathcal{F}^{\text{pro}}(X_\infty) := \varinjlim_{\lambda \in \Lambda} \left\{ \mathcal{F}(X_\lambda), \pi_{\lambda\mu}^* : \mathcal{F}(X_\lambda) \rightarrow \mathcal{F}(X_\mu) (\lambda < \mu) \right\},$$

which also *may not belong to* the category  $\mathcal{C}'$ . Although  $\mathcal{F}^{\text{pro}}(X_\infty)$  is in general an abstract object assigned to the projective limit  $X_\infty$ , in the case of constructible functions,  $F^{\text{pro}}(X_\infty)$  can be treated as a group of functions on  $X_\infty$  as observed above and more



detailed discussion on  $F^{\text{pro}}(X_\infty)$  will be given in connection with motivic measures in a later section. In general situations, this abstract object is sufficient.

Now, instead of considering  $F^{\text{pro}}(X_\infty)$  of an arbitrary proalgebraic variety  $X_\infty$ , we first consider it for the infinite countable product  $X^{\mathbb{N}}$  of a complex algebraic variety  $X$  as a simple model case.

For each projection  $\pi_{(n-1)n} : X^n \rightarrow X^{n-1}$ , the pullback homomorphism  $\pi_{(n-1)n}^* : F(X^{n-1}) \rightarrow F(X^n)$  is the multiplication by the characteristic function  $\mathbb{1}_X$  of the last factor  $X$ , i.e.,

$$\pi_{(n-1)n}^*(\alpha) = \alpha \times \mathbb{1}_X,$$

where  $(\alpha \times \mathbb{1}_X)(y, x) := \alpha(y)\mathbb{1}_X(x) = \alpha(y)$ . Then, using the cross product formula  $c_*(\delta \times \omega) = c_*(\delta) \times c_*(\omega)$  of the Chern–Schwartz–MacPherson class  $c_*$  (see [Kw] and also cf. [KY]), we get the following commutative diagram

$$\begin{array}{ccc} F(X^{n-1}) & \xrightarrow{\times \mathbb{1}_X} & F(X^n) \\ c_* \downarrow & & \downarrow c_* \\ H_*(X^{n-1}) & \xrightarrow{\times c_*(X)} & H_*(X^n). \end{array}$$

So, if we set

$$H_*^{\text{pro}}(X^{\mathbb{N}}) := \varinjlim_n \left\{ \times c_*(X) : H_*(X^{n-1}) \rightarrow H_*(X^n) \right\},$$

then we have a *proalgebraic Chern–Schwartz–MacPherson class homomorphism*:

$$c_*^{\text{pro}} : F^{\text{pro}}(X^{\mathbb{N}}) \rightarrow H_*^{\text{pro}}(X^{\mathbb{N}})$$

Let us look at the “0-dimensional component” of this homomorphism, i.e.,  $\int_{X^{\mathbb{N}}} c_*^{\text{pro}}$ . Namely, we consider the following commutative diagram for each  $n$

$$(3.2) \quad \begin{array}{ccc} F(X^{n-1}) & \xrightarrow{\times \mathbb{1}_X} & F(X^n) \\ \chi \downarrow & & \downarrow \chi \\ \mathbb{Z} & \xrightarrow{\times \chi(X)} & \mathbb{Z}. \end{array}$$

and we take its inductive limit, which gives us the *proalgebraic Euler–Poincaré characteristic (homomorphism)*

$$\chi^{\text{pro}} : F^{\text{pro}}(X^{\mathbb{N}}) \rightarrow \varinjlim_n \left\{ \times \chi(X) : \mathbb{Z} \rightarrow \mathbb{Z} \right\}.$$

It is clear from the definition that for  $\alpha_n \in F(X^n)$  we have  $\chi^{\text{pro}}([\alpha_n]) = [\chi(\alpha_n)]$ . To describe  $[\chi(\alpha_n)]$ , i.e.,  $\chi^{\text{pro}}$ , more explicitly or down-to-earth, we recall the following lemma:

**Lemma (3.3).** *Let  $p$  be a non-zero integer. For each positive integer  $n$ , let  $G_n = \mathbb{Z}$  be the integers and  $\pi_{n(n+1)} : G_n \rightarrow G_{n+1}$  be the homomorphism defined by multiplication by  $p$ , i.e.,  $\pi_{n(n+1)}(m) = pm$ . Then the inductive limit  $\varinjlim_n G_n$  is isomorphic to the abelian group of all rationals of the form  $\frac{m}{p^n}$ , for integers  $m$  and  $n$ , which shall be denoted by  $\mathbb{Z} \left[ \frac{1}{p} \right]$ . If  $p = 0$ , then the inductive limit  $\varinjlim_n G_n = 0$ .  $\square$*

This seems to be a well-known fact (cf. [Munk, §73, Example 4, p. 435]) and this simple fact turns out to be a key in what follows. For the sake of later use, we give a quick proof. For each  $n$ , let  $\phi^n : G_n \rightarrow \mathbb{Z} \left[ \frac{1}{p} \right]$  be defined by  $\phi^n(m) = \frac{m}{p^{n-1}}$ . Then we have the commutative diagram

$$\begin{array}{ccc} G_n & \xrightarrow{\pi_{n(n+1)}} & G_{n+1} \\ & \searrow \phi^n & \swarrow \phi^{n+1} \\ & \mathbb{Z} \left[ \frac{1}{p} \right] & \end{array}$$

It follows from standard facts on inductive systems and inductive limits that all these commutative diagrams give rise to the unique homomorphism

$$\Phi : \varinjlim_n \phi^n : \varinjlim_n G_n \rightarrow \mathbb{Z} \left[ \frac{1}{p} \right] \quad \text{described by} \quad \Phi \left( \sum_k \rho^k(m_k) \right) = \sum_k \frac{m_k}{p^{k-1}},$$

which turns out to be surjective and injective.

There are of course infinitely many other isomorphisms between  $\varinjlim_n G_n$  and  $\mathbb{Z} \left[ \frac{1}{p} \right]$ . Indeed, let for any integer  $w$ , we define  $\phi_w^n : G_n \rightarrow \mathbb{Z} \left[ \frac{1}{p} \right]$  by  $\phi_w^n(m) = \frac{m}{p^{n-1+w}}$ . Then for the integer  $w$  we get an isomorphism

$$\Phi_w = \varinjlim_n \phi_w^n : \varinjlim_n G_n \rightarrow \mathbb{Z} \left[ \frac{1}{p} \right] \quad \text{defined by} \quad \Phi_w \left( \sum_k \rho^k(m_k) \right) = \sum_k \frac{m_k}{p^{k-1+w}}.$$

When  $w = 0$ , the above  $\Phi_0 = \Phi = \varinjlim_n \phi^n : \varinjlim_n G_n \rightarrow \mathbb{Z} \left[ \frac{1}{p} \right]$  is called the *canonical isomorphism*.

Using this lemma we have the following proalgebraic version of the Euler–Poincaré characteristic homomorphism  $\chi : F(V) \rightarrow \mathbb{Z}$ :

**Theorem (3.4).** *Let  $X$  be a complex algebraic variety and we assume that  $\chi(X) \neq 0$ . The canonical Euler–Poincaré (pro)characteristic homomorphism*

$$\chi^{\text{pro}} : F^{\text{pro}}(X^{\mathbb{N}}) \rightarrow \mathbb{Z} \left[ \frac{1}{\chi(X)} \right]$$

is described by

$$\chi^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\chi(\alpha_n)}{\chi(X)^{n-1}}$$

where  $\alpha_n \in F(X^n)$ .  $\square$

In particular, if  $\mathbb{1}_{X^{\mathbb{N}}}$  denotes the characteristic function on the proalgebraic variety  $X^{\mathbb{N}}$ , hence  $\mathbb{1}_{X^{\mathbb{N}}} = [\mathbb{1}_{X^k}]$  for any  $k$ , we have

$$\chi^{\text{pro}}(\mathbb{1}_{X^{\mathbb{N}}}) = \frac{\chi(\mathbb{1}_{X^k})}{\chi(X)^{k-1}} = \frac{\chi(X)^k}{\chi(X)^{k-1}} = \chi(X)$$

which is called *the canonical Euler–Poincaré (pro)characteristic* of the proalgebraic variety  $X^{\mathbb{N}}$  and simply denoted by  $\chi^{\text{pro}}(X^{\mathbb{N}})$ .

This simple and naïve observation, which looked nonsensical or meaningless at the beginning, was the very start of the present work. Note that  $c_*^{\text{pro}}(X^{\mathbb{N}}) := c_*^{\text{pro}}(\mathbb{1}_{X^{\mathbb{N}}}) = [c_*(X)]$ , which is of course equal to  $[c_*(X^k)]$  for any positive integer  $k$ .

As pointed out above, for any integer  $w$

$$\chi_w^{\text{pro}} \left( \sum_n [\alpha_n] \right) := \sum_n \frac{\chi(\alpha_n)}{\chi(X)^{n-1+w}}$$

is also an Euler–Poincaré (pro)characteristic homomorphism. Unless stated otherwise, we consider only the above canonical one.

Theorem (3.4) can be generalized to the following:

**Theorem (3.5).** *Let  $X_{\infty} = \varprojlim_{n \in \mathbb{N}} X_n$  be a proalgebraic variety such that for each  $n$  the structure morphism  $\pi_{n(n+1)} : X_{n+1} \rightarrow X_n$  satisfies the condition that the Euler–Poincaré characteristics of the fibers of  $\pi_{n,n+1}$  are non-zero (which implies the surjectivity of the morphism  $\pi_{n(n+1)} : X_{n+1} \rightarrow X_n$  is a locally trivial fiber bundle with fiber variety being  $F_n$  and  $\chi(F_n) \neq 0$ ). Let us denote the constant Euler–Poincaré characteristic of the fibers of the morphism  $\pi_{n(n+1)} : X_{n+1} \rightarrow X_n$  by  $\chi_n$  and we set  $\chi_0 := 1$ .*

(i) *The canonical Euler–Poincaré (pro)characteristic homomorphism*

$$\chi^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow \mathbb{Q}$$

is described by

$$\chi^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\chi(\alpha_n)}{\chi_0 \cdot \chi_1 \cdot \chi_2 \cdots \chi_{n-1}}.$$

(ii) *In particular, if the Euler–Poincaré characteristics  $\chi_n$  are all the same, say  $\chi_n = \chi$  for any  $n$ , then the canonical Euler–Poincaré (pro)characteristic homomorphism  $\chi^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow \mathbb{Q}$  is described by*

$$\chi^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\chi(\alpha_n)}{\chi^{n-1}}.$$

In this special case, the target ring  $\mathbb{Q}$  can be replaced by the ring  $\mathbb{Z} \left[ \frac{1}{\chi} \right]$ , the non-canonical one  $\chi_w^{\text{pro}}$  is described by

$$\chi_w^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\chi(\alpha_n)}{\chi^{n-1+w}}.$$

(iii) In particular, if  $\chi_n = \chi(\mathbb{P}^n)$  for each  $n$  or  $F_n = \mathbb{P}^n$  the complex projective space of dimension  $n$  for each  $n$  in the above example, then the canonical Euler–Poincaré (pro)characteristic homomorphism  $\chi^{\text{pro}}$  is surjective.  $\square$

In particular, by considering the characteristic function  $\mathbb{1}_{X_\infty} = [\mathbb{1}_{X_n}]$  for any  $n$ ,

$$\chi^{\text{pro}}(X_\infty) := \chi^{\text{pro}}(\mathbb{1}_{X_\infty}) = \frac{\chi(X_1) \cdot \chi_1 \cdot \chi_2 \cdots \chi_{n-1}}{\chi_1 \cdot \chi_2 \cdots \chi_{n-1}} = \chi(X_1),$$

which is called the canonical Euler–Poincaré (pro)characteristic of the above proalgebraic variety  $X_\infty$ .

*Proof of Theorem (3.5).* For a morphism  $f : X \rightarrow Y$  whose fibers all have the same non-zero Euler–Poincaré (pro)characteristic, denoted by  $\chi_f$ , we get the commutative diagram

$$(3.5.1) \quad \begin{array}{ccc} F(Y) & \xrightarrow{f^*} & F(X) \\ \chi \downarrow & & \downarrow \chi \\ \mathbb{Z} & \xrightarrow{\times \chi_f} & \mathbb{Z}. \end{array}$$

To see the commutativity of this diagram, we use the fact that for any constructible function  $\beta \in F(X)$  we have

$$\chi(\beta) = \chi(f_*\beta),$$

which follows from the naturality of the Chern–Schwartz–MacPherson class  $c_*$ . Indeed, using the above fact, we can show that for any characteristic function  $\mathbb{1}_W \in F(Y)$  we have

$$\begin{aligned} \chi(f^*\mathbb{1}_W) &= \chi(f_*f^*\mathbb{1}_W) \\ &= \chi(\chi_f \cdot \mathbb{1}_W) \\ &= \chi_f \cdot \chi(\mathbb{1}_W). \end{aligned}$$

Therefore the above diagram (3.5.1) commutes. Thus the theorem follows from the following commutative diagram for each  $n$ ;

$$(3.5.2) \quad \begin{array}{ccc} F(X_n) & \xrightarrow{\pi_{n(n+1)}^*} & F(X_{n+1}) \\ \chi \downarrow & & \downarrow \chi \\ \mathbb{Z} & \xrightarrow{\times \chi_n} & \mathbb{Z} \end{array}$$

and the following generalized version of Lemma (3.3):

**Lemma (3.5.3).** *For each positive integer  $n$ , let  $G_n = \mathbb{Z}$  be the integers and  $\pi_{n,n+1} : G_n \rightarrow G_{n+1}$  be the homomorphism defined by multiplication by a non-zero integer  $p_n$ , i.e.,  $\pi_{n,n+1}(m) = mp_n$ . Then there exists a unique (injective) homomorphism*

$$\Psi : \varinjlim_n G_n \rightarrow \mathbb{Q}$$

such that the following diagram commutes

$$\begin{array}{ccc} & G_n & \\ \rho^n \swarrow & & \searrow \times \frac{1}{p_0 p_1 \cdots p_{n-1}} \\ \varinjlim_n G_n & \xrightarrow{\Psi} & \mathbb{Q}. \end{array}$$

Here we set  $p_0 := 1$ . And  $\Psi$  is described by

$$\Psi \left( \sum_n \rho^n(r_n) \right) = \sum_n \frac{r_n}{p_0 p_1 \cdots p_{n-1}}.$$

Thus we get (i) and (ii). In the case of (iii), the injective homomorphism  $\Psi : \varinjlim_n G_n \rightarrow \mathbb{Q}$  becomes surjective, thus we get (iii).  $\square$

Let the situation be as in Theorem (3.5). Let  $\widehat{F^{\text{pro}}}(X_\infty)$  denote the abelian group of formal infinite sums  $\sum_{n=1}^{\infty} [\alpha_n]$  such that its proalgebraic Euler–Poincaré characteristic  $\sum_{n=1}^{\infty} \chi^{\text{pro}}([\alpha_n])$  converges. Thus  $\widehat{F^{\text{pro}}}(X_\infty)$  can be interpreted as a *completion* of  $F^{\text{pro}}(X_\infty)$ . With this definition we can get the following theorem, which is impossible in the usual algebraic geometry.

**Theorem (3.6).** *For the infinite product space*

$$\prod_{n=1}^{\infty} \mathbb{P}^n := \mathbb{P}^1 \times \mathbb{P}^2 \times \cdots \times \mathbb{P}^n \times \cdots$$

(which is a proalgebraic variety), the proalgebraic Euler–Poincaré characteristic homomorphism

$$\widehat{\chi^{\text{pro}}} : \widehat{F^{\text{pro}}} \left( \prod_{n=1}^{\infty} \mathbb{P}^n \right) \rightarrow \mathbb{R} \quad \text{defined by} \quad \widehat{\chi^{\text{pro}}} \left( \sum_{n=1}^{\infty} [\alpha_n] \right) = \sum_{n=1}^{\infty} \chi^{\text{pro}}([\alpha_n])$$

is surjective.  $\square$

Up to now, to get our results, we use the commutative diagrams (3.2) and (3.5.2), i.e., the constancy of the Euler–Poincaré characteristics of all the fibers of each structure morphism. Thus, in order to consider the proalgebraic Chern–Schwartz–MacPherson

classes or the Euler–Poincaré characteristics of proalgebraic varieties from the viewpoint of inductive limits, structure morphisms constituting projective systems must have such a strong requirement.

In fact, Theorem (3.5) can be extended to  $c_*^{\text{pro}}$ . To state and prove such a theorem, we need to appeal to the Bivariant Theory introduced by William Fulton and Robert MacPherson [FM], in particular a bivariant Chern class [Br]. So, we quickly recall only necessary ingredients of the Bivariant Theory for our use.

A bivariant theory  $\mathbb{B}$  on a category  $\mathcal{C}$  with values in the category of abelian groups is an assignment to each morphism

$$X \xrightarrow{f} Y$$

in the category  $\mathcal{C}$  a graded abelian group

$$\mathbb{B}(X \xrightarrow{f} Y)$$

which is equipped with the following three basic operations:

(Product operations): For morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$ , the product operation

$$\bullet : \mathbb{B}(X \xrightarrow{f} Y) \otimes \mathbb{B}(Y \xrightarrow{g} Z) \rightarrow \mathbb{B}(X \xrightarrow{gf} Z)$$

is defined.

(Pushforward operations): For morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  with  $f$  proper, the pushforward operation

$$f_\star : \mathbb{B}(X \xrightarrow{gf} Z) \rightarrow \mathbb{B}(Y \xrightarrow{g} Z)$$

is defined.

(Pullback operations): For a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation

$$g^\star : \mathbb{B}(X \xrightarrow{f} Y) \rightarrow \mathbb{B}(X' \xrightarrow{f'} Y')$$

is defined.

And these three operations are required to satisfy the seven compatibility axioms (see [FM, Part I, §2.2] for details).

Let  $\mathbb{B}, \mathbb{B}'$  be two bivariant theories on a category  $\mathcal{C}$ . Then a *Grothendieck transformation* from  $\mathbb{B}$  to  $\mathbb{B}'$

$$\gamma : \mathbb{B} \rightarrow \mathbb{B}'$$

is a collection of homomorphisms

$$\mathbb{B}(X \rightarrow Y) \rightarrow \mathbb{B}'(X \rightarrow Y)$$

for a morphism  $X \rightarrow Y$  in the category  $\mathcal{C}$ , which preserves the above three basic operations:

- (i)  $\gamma(\alpha \bullet_{\mathbb{B}} \beta) = \gamma(\alpha) \bullet_{\mathbb{B}'} \gamma(\beta)$ ,
- (ii)  $\gamma(f_{\star} \alpha) = f_{\star} \gamma(\alpha)$ , and
- (iii)  $\gamma(g^{\star} \alpha) = g^{\star} \gamma(\alpha)$ .

A bivariant theory unifies both a covariant theory and a contravariant theory in the following sense:  $B_*(X) := \mathbb{B}(X \rightarrow pt)$  and  $B^*(X) := \mathbb{B}(X \xrightarrow{id} X)$  become a covariant functor and a contravariant functor, respectively. And a Grothendieck transformation  $\gamma : \mathbb{B} \rightarrow \mathbb{B}'$  induces natural transformations  $\gamma_* : B_* \rightarrow B'_*$  and  $\gamma^* : B^* \rightarrow B'^*$ . Note also that if we have a Grothendieck transformation  $\gamma : \mathbb{B} \rightarrow \mathbb{B}'$ , then via a bivariant class  $b \in \mathbb{B}(X \xrightarrow{f} Y)$  we get the commutative diagram

$$\begin{array}{ccc} B_*(Y) & \xrightarrow{\gamma_*} & B'_*(Y) \\ b \bullet \downarrow & & \downarrow \gamma(b) \bullet \\ B_*(X) & \xrightarrow{\gamma_*} & B'_*(X). \end{array}$$

This is called *the Verdier-type Riemann–Roch associated to the bivariant class  $b$* .

Fulton–MacPherson’s bivariant group  $\mathbb{F}(X \xrightarrow{f} Y)$  of constructible functions consists of all the constructible functions on  $X$  which satisfy the local Euler condition with respect to  $f$ . Here a constructible function  $\alpha \in F(X)$  is said to satisfy the *local Euler condition with respect to  $f$*  if for any point  $x \in X$  and for any local embedding  $(X, x) \rightarrow (\mathbf{C}^N, 0)$  the equality  $\alpha(x) = \chi(B_{\epsilon} \cap f^{-1}(z); \alpha)$  holds, where  $B_{\epsilon}$  is a sufficiently small open ball of the origin 0 with radius  $\epsilon$  and  $z$  is any point close to  $f(x)$  (cf. [Br], [Sa2]). In particular, if  $\mathbb{1}_f := \mathbb{1}_X$  belongs to the bivariant group  $\mathbb{F}(X \xrightarrow{f} Y)$ , then the morphism  $f : X \rightarrow Y$  is called *an Euler morphism*. And any constructible function in the bivariant group  $\mathbb{F}(X \xrightarrow{f} Y)$  is called *a bivariant constructible function* to emphasize the bivariantness.

The three operations on  $\mathbb{F}$  are defined as follows:

- (i) the product operation  $\bullet : \mathbb{F}(X \xrightarrow{f} Y) \otimes \mathbb{F}(Y \xrightarrow{g} Z) \rightarrow \mathbb{F}(X \xrightarrow{gf} Z)$  is defined by

$$\alpha \bullet \beta := \alpha \cdot f^* \beta$$

,

- (ii) the pushforward operation  $f_{\star} : \mathbb{F}(X \xrightarrow{gf} Z) \rightarrow \mathbb{F}(Y \xrightarrow{g} Z)$  is the usual pushforward  $f_*$ , i.e.,

$$f_{\star}(\alpha)(y) := \int c_*(\alpha|_{f^{-1}}),$$

- (iii) for a fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the pullback operation  $g^\star : \mathbb{F}(X \xrightarrow{f} Y) \rightarrow \mathbb{F}(X' \xrightarrow{f'} Y')$  is the functional pullback  $g'^*$ , i.e.,

$$g^\star(\alpha)(x') := \alpha(g'(x')).$$

Note that  $\mathbb{F}(X \xrightarrow{\text{id}_X} X)$  consists of all locally constant functions and  $\mathbb{F}(X \rightarrow pt) = F(X)$ . As a corollary of this observation, we have

**Proposition (3.7).** *For any bivariant constructible function  $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ , the Euler–Poincaré characteristic  $\chi(f^{-1}(y); \alpha) = \int c_*(\alpha|_{f^{-1}(y)})$  of  $\alpha$  restricted to each fiber  $f^{-1}(y)$  is locally constant, i.e., constant along connected components of the base variety  $Y$ . In particular, if  $f : X \rightarrow Y$  is an Euler morphism, then the Euler–Poincaré characteristic of the fibers are locally constant.*

Note that locally trivial fiber bundles are Euler, but not vice versa.

Let  $\mathbb{H}$  be Fulton–MacPherson’s bivariant homology theory, constructed from the cohomology theory. For a morphism  $f : X \rightarrow Y$ , choose a morphism  $\phi : X \rightarrow \mathbb{R}^n$  such that  $\Phi := (f, \phi) : X \rightarrow Y \times \mathbb{R}^n$  is a closed embedding. Then the  $i$ -th bivariant homology group  $\mathbb{H}^i(X \xrightarrow{f} Y)$  is defined by

$$\mathbb{H}^i(X \xrightarrow{f} Y) := H^{i+n}(Y \times \mathbb{R}^n, Y \times \mathbb{R}^n \setminus X_\phi),$$

where  $X_\phi$  is defined to be the image of the morphism  $\Phi = (f, \phi)$ . The definition is independent of the choice of  $\phi$ . Note that instead of taking the Euclidean space  $\mathbb{R}^n$  we can take a manifold  $M$  so that  $i : X \rightarrow M$  is a closed embedding and then consider the graph embedding  $f \times i : X \rightarrow Y \times M$ . See [FM, §3.1] for more details of  $\mathbb{H}$ . In particular, note that if  $Y$  is a point  $pt$ ,  $\mathbb{H}(X \rightarrow pt)$  is isomorphic to the homology group  $H_*(X)$  of the source variety  $X$ .

W. Fulton and R. MacPherson conjectured or posed as a question the existence of a so-called bivariant Chern class and J.-P. Brasselet [Br] solved it:

**Theorem (3.8).** *(J.-P. Brasselet) On the category of complex analytic varieties and cellular morphisms, there exists a Grothendieck transformation*

$$\gamma^{\text{Br}} : \mathbb{F} \rightarrow \mathbb{H}$$

satisfying the normalization condition that  $\gamma^{\text{Br}}(\mathbb{1}_\pi) = c(TX) \cap [X]$  for  $X$  smooth, where  $\pi : X \rightarrow pt$  and  $\mathbb{1}_\pi = \mathbb{1}_X$ .

Note that for a morphism  $f : X \rightarrow pt$  from a variety  $X$  to a point  $pt$ ,  $\gamma^{\text{Br}} : \mathbb{F}(X \rightarrow pt) \rightarrow \mathbb{H}(X \rightarrow pt)$  is nothing but the original Chern–Schwartz–MacPherson class  $c_* : F(X) \rightarrow H_*(X)$ .

**Corollary (3.9).** *(Verdier-type Riemann–Roch for Chern class) For a bivariant constructible function  $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$  we have the following commutative diagram:*

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ \alpha \bullet_{\mathbb{F}} = \alpha \cdot f^* \downarrow & & \downarrow \gamma^{\text{Br}}(\alpha) \bullet_{\mathbb{H}} \\ F(X) & \xrightarrow{c_*} & H_*(X). \end{array}$$



In particular, for an Euler morphism we have the following diagram:

$$\begin{array}{ccc} F(Y) & \xrightarrow{c_*} & H_*(Y) \\ \mathbf{1}_f \bullet_{\mathbb{R}} = f^* \downarrow & & \downarrow \gamma^{\text{Br}}(\mathbf{1}_f) \bullet_{\mathbb{H}} \\ F(X) & \xrightarrow{c_*} & H_*(X). \end{array}$$

(The homomorphism  $\gamma^{\text{Br}}(\mathbf{1}_f) \bullet_{\mathbb{H}}$  shall be denoted by  $f^{**}$ .)

For a more generalized Verdier-type Riemann–Roch theorem for Chern–Schwartz–MacPherson class, see Jörg Schürmann’s recent article [Sch1].

**Theorem (3.10).** *Let  $X_{\infty} = \varinjlim_{\lambda \in \Lambda} X_{\lambda}$  be a proalgebraic variety such that for each  $\lambda < \mu$  the structure morphism  $\pi_{\lambda\mu} : X_{\mu} \rightarrow X_{\lambda}$  is an Euler proper morphism (hence surjective) of topologically connected algebraic varieties with the constant Euler-Poincaré characteristic  $\chi_{\lambda\mu}$  of the fiber of the morphism  $\pi_{\lambda\mu}$  being non-zero. Let  $H_*^{\text{pro}}(X_{\infty})$  be the inductive limit of the inductive system  $\{\pi_{\lambda\mu}^{**} : H_*(X_{\lambda}) \rightarrow H_*(X_{\mu})\}$ .*

*There exists a proalgebraic Chern–Schwartz–MacPherson class homomorphism*

$$c_*^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow H_*^{\text{pro}}(X_{\infty}) \quad \text{defined by} \quad c_*^{\text{pro}} \left( \sum_{\lambda} [\alpha_{\lambda}] \right) = \sum_{\lambda} \rho^{\lambda}(c_*(\alpha_{\lambda})).$$

In particular, the proalgebraic integration becomes the following

$$\chi^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow \varinjlim_{\Lambda} \{ \times \chi_{\lambda\mu} : \mathbb{Z} \rightarrow \mathbb{Z} \}.$$

(Here we do not know an explicit description of the inductive limit  $\varinjlim_{\Lambda} \{ \times \chi_{\lambda\mu} : \mathbb{Z} \rightarrow \mathbb{Z} \}$  like Lemma (3.5.3).)  $\square$

In this general situation we have that

$$c_*^{\text{pro}}(X_{\infty}) = [c_*(X_{\lambda})] \quad \text{and} \quad \chi^{\text{pro}}(X_{\infty}) = [\chi(X_{\lambda})]$$

for any  $\lambda \in \Lambda$ , but we cannot give a more explicit description like in the case when  $\Lambda = \mathbb{N}$ .

What we have done so far is the proalgebraic Chern–Schwartz–MacPherson class homomorphism, and our eventual problem is whether one can capture this homomorphism as a *natural transformation* as in the original Chern–Schwartz–MacPherson class.

First, as a trial, we consider a very simple promorphism

$$f : X^{\mathbb{N}} \rightarrow Y^{\mathbb{N}}$$

which is one of the important objects in Gromov’s papers [Grom 1, Grom 2]. Namely, we consider the naturality of  $c_*^{\text{pro}} : F^{\text{pro}}(X^{\mathbb{N}}) \rightarrow H_*^{\text{pro}}(X^{\mathbb{N}})$ . Even in this simple case it turns out that we cannot expect a reasonable result. For example, for a promorphism  $f : X^{\mathbb{N}} \rightarrow$

$Y^{\mathbb{N}}$  consider the the projective limit of  $f^n := \underbrace{f \times \cdots \times f}_{n \text{ times}} : X^n \rightarrow Y^n$  for a morphism  $f : X \rightarrow Y$ . For more nontrivial examples involving subtle combinatoric natures of graphs, see [Grom 1]. Even in the above simplest case, one cannot get a reasonable solution even if we consider the naturality of the homomorphism  $\chi^{\text{pro}} : F^{\text{pro}}(X^{\mathbb{N}}) \rightarrow \mathbb{Z} \left[ \frac{1}{\chi(X)} \right]$ . Our first simple-minded and hasty answer was that if we just took the “denominator changing” homomorphism  $D_{X/Y} : \mathbb{Z} \left[ \frac{1}{\chi(X)} \right] \rightarrow \mathbb{Z} \left[ \frac{1}{\chi(Y)} \right]$  defined by

$$D_{X/Y} \left( \sum_k \frac{a_k}{\chi(X)^k} \right) := \sum_k \frac{a_k}{\chi(Y)^k}$$

then we would get the following commutative diagram, i.e., the naturality of  $\chi^{\text{pro}}$ :

$$\begin{array}{ccc} F^{\text{pro}}(X^{\mathbb{N}}) & \xrightarrow{f_*} & F^{\text{pro}}(Y^{\mathbb{N}}) \\ \chi^{\text{pro}} \downarrow & & \downarrow \chi^{\text{pro}} \\ \mathbb{Z} \left[ \frac{1}{\chi(X)} \right] & \xrightarrow{D_{X/Y}} & \mathbb{Z} \left[ \frac{1}{\chi(Y)} \right]. \end{array}$$

Which is “because” of the following computation:

$$\begin{aligned} \chi^{\text{pro}} \left( f_* \left( \sum_n [\alpha_n] \right) \right) &= \sum_n \frac{\chi((f^n)_* \alpha_n)}{\chi(Y)^{n-1}} \\ &= \sum_n \frac{\chi(\alpha_n)}{\chi(Y)^{n-1}} \quad (\text{since } \chi((f^n)_* \alpha_n) = \chi(\alpha_n)) \\ &= D_{X/Y} \left( \sum_n \frac{\chi(\alpha_n)}{\chi(X)^{n-1}} \right) \\ &= D_{X/Y} \left( \chi^{\text{pro}} \left( \sum_n [\alpha_n] \right) \right). \end{aligned}$$

However, this “denominator changing” homomorphism  $D_{X/Y} : \mathbb{Z} \left[ \frac{1}{\chi(X)} \right] \rightarrow \mathbb{Z} \left[ \frac{1}{\chi(Y)} \right]$  is not well-defined, since it is not independent of the expression of the rational.

So, to get a reasonable result, we have to restrict ourselves to some special promorphisms.

If the commutative diagram

$$\begin{array}{ccc} Y_{\xi(\mu)} & \xrightarrow{f_\mu} & X_\mu \\ \rho_{\xi(\lambda)\xi(\mu)} \downarrow & & \downarrow \pi_{\lambda\mu} \\ Y_{\xi(\lambda)} & \xrightarrow{f_\lambda} & X_\lambda, \end{array}$$

is a fiber square, then we call the pro-morphism  $\{f_\lambda : Y_{\xi(\lambda)} \rightarrow X_\lambda\}$  a *fiber-square pro-morphism*, abusing words.

Let  $\{f_\lambda : Y_{\xi(\lambda)} \rightarrow X_\lambda\}$  be a fiber-square pro-morphism between two pro-algebraic varieties with Euler morphisms of possibly singular terms  $X'_\lambda$ s and  $Y'_\gamma$ s. Then our question is whether the following diagram is commutative or not:

$$\begin{array}{ccc} F^{\text{pro}}(Y_\infty) & \xrightarrow{c_*^{\text{pro}}} & H_*^{\text{pro}}(Y_\infty) \\ f_{\infty*} \downarrow & & \downarrow f_{\infty*} \\ F^{\text{pro}}(X_\infty) & \xrightarrow{c_*^{\text{pro}}} & H_*^{\text{pro}}(X_\infty). \end{array}$$

To prove the commutativity of this diagram, it suffices to show the commutativity of the following diagrams:

$$\begin{array}{ccccc} & & F(Y_{\xi(\lambda)}) & \xrightarrow{f_{\lambda*}} & F(X_\lambda) \\ & \swarrow c_* & \downarrow & & \swarrow c_* \\ H_*(Y_{\xi(\lambda)}) & \xrightarrow{f_{\lambda*}} & H_*(X_\lambda) & & \downarrow \pi_{\lambda\mu}^* \\ & \searrow \rho_{\xi(\lambda)\xi(\mu)}^* & \downarrow & & \downarrow \pi_{\lambda\mu}^{**} \\ & & F(Y_{\xi(\mu)}) & \xrightarrow{f_{\mu*}} & F(X_\mu) \\ \rho_{\xi(\lambda)\xi(\mu)}^{**} \downarrow & & \swarrow c_* & & \swarrow c_* \\ H_*(Y_{\xi(\mu)}) & \xrightarrow{f_{\mu*}} & H_*(X_\mu) & & \end{array}$$

The commutativity of the top and bottom squares is due to the naturality of the Chern–Schwartz–MacPherson class [M], the commutativity of the right and left squares is due to the above Corollary (3.9), and the commutativity of the outer big square is due to the following fact: For the fiber square

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ f' \downarrow & & \downarrow f \\ Y' & \xrightarrow{g} & Y, \end{array}$$

the following diagram commutes (e.g., see [Er, Proposition 3.5], [FM, Axiom  $(A_{23})$ ]):

$$\begin{array}{ccc} F(Y') & \xrightarrow{f'^*} & F(X') \\ g_* \downarrow & & \downarrow g'_* \\ F(Y) & \xrightarrow{f_*} & F(X). \end{array}$$

Thus it remains to see only the commutativity of the inner small square, i.e., for any  $y \in H_*(Y_{\xi(\lambda)})$

$$\pi_{\lambda\mu}^{**}(f_{\lambda*}(y)) = f_{\mu*}(\rho_{\xi(\lambda)\xi(\mu)}^{**}(y)),$$

which is more precisely

$$\gamma^{\text{Br}}(\mathbb{1}_{\pi_{\lambda\mu}}) \bullet_{\mathbb{H}} f_{\lambda\star}(y) = f_{\mu\star}(\gamma^{\text{Br}}(\mathbb{1}_{\rho_{\xi(\lambda)\xi(\mu)}}) \bullet_{\mathbb{H}} y).$$

Since  $\mathbb{1}_{\rho_{\xi(\lambda)\xi(\mu)}} = f_{\lambda}^{\star} \mathbb{1}_{\pi_{\lambda\mu}}$  and the Grothendieck transformation  $\gamma^{\text{Br}} : \mathbb{F} \rightarrow \mathbb{H}$  is compatible with the pullback operation, the above equality becomes

$$\gamma^{\text{Br}}(\mathbb{1}_{\pi_{\lambda\mu}}) \bullet_{\mathbb{H}} f_{\lambda\star}(y) = f_{\mu\star}(f_{\lambda}^{\star} \gamma^{\text{Br}}(\mathbb{1}_{\pi_{\lambda\mu}}) \bullet_{\mathbb{H}} y).$$

And it turns out that this equality is nothing but *the projection formula* of the Bivariant Theory [FM, §2.2, (A<sub>123</sub>)] for the following diagram and for the bivariant homology theory  $\mathbb{H}$ :

$$\begin{array}{ccc} Y_{\xi(\mu)} & \xrightarrow{f_{\mu}} & X_{\mu} \\ \rho_{\xi(\lambda)\xi(\mu)} \downarrow & & \downarrow \pi_{\lambda\mu} \\ Y_{\xi(\lambda)} & \xrightarrow{f_{\lambda}} & X_{\lambda} \longrightarrow pt. \end{array}$$

Thus we obtain the following theorem.

**Theorem (3.11).** *Let  $\{f_{\lambda} : Y_{\xi(\lambda)} \rightarrow X_{\lambda}\}$  be a fiber-square pro-morphism between two pro-algebraic varieties with structure morphisms being Euler morphisms. Then we have the following commutative diagram:*

$$\begin{array}{ccc} F^{\text{pro}}(Y_{\infty}) & \xrightarrow{c_*^{\text{pro}}} & H_*^{\text{pro}}(Y_{\infty}) \\ f_{\infty*} \downarrow & & \downarrow f_{\infty*} \\ F^{\text{pro}}(X_{\infty}) & \xrightarrow{c_*^{\text{pro}}} & H_*^{\text{pro}}(X_{\infty}). \end{array}$$

□

Following the above construction, similarly we can get a proalgebraic version of the Riemann–Roch theorem, i.e., the Baum–Fulton–MacPherson’s Riemann–Roch  $\tau_* : \mathbf{K}_0 \rightarrow H_*\mathbb{Q}$  constructed in [BFM].

**Theorem (3.12).** *Let  $\{f_{\lambda} : Y_{\xi(\lambda)} \rightarrow X_{\lambda}\}$  be a fiber-square pro-morphism between two pro-algebraic varieties with structure morphisms being proper local complete intersection morphisms. Then we have the following commutative diagram:*

$$\begin{array}{ccc} \mathbf{K}_0^{\text{pro}}(Y_{\infty}) & \xrightarrow{\tau_*^{\text{pro}}} & H_*^{\text{pro}}(Y_{\infty}) \\ f_{\infty*} \downarrow & & \downarrow f_{\infty*} \\ \mathbf{K}_0^{\text{pro}}(X_{\infty}) & \xrightarrow{\tau_*^{\text{pro}}} & H_*^{\text{pro}}(X_{\infty}). \end{array}$$

□

These results lead us to much more general theorems. First we introduce the following notion. For a morphism  $f : X \rightarrow Y$  and a bivariant class  $b \in \mathbb{B}(X \xrightarrow{f} Y)$ , the pair  $(f; b)$  is called a *bivariant-class-equipped morphism* and we just express  $(f; b) : X \rightarrow Y$ . Let  $\{(\pi_{\lambda\mu}; b_{\lambda\mu}) : X_\mu \rightarrow X_\lambda\}$  be a system of bivariant-class-equipped morphisms. If a system  $\{b_{\lambda\mu}\}$  of bivariant classes satisfies that

$$b_{\mu\nu} \bullet b_{\lambda\mu} = b_{\lambda\nu} \quad (\lambda < \mu < \nu)$$

then we call the system a *projective system of bivariant classes*, abusing words. If  $\{\pi_{\lambda\mu} : X_\mu \rightarrow X_\lambda\}$  and  $\{b_{\lambda\mu}\}$  are projective systems, then the system  $\{(\pi_{\lambda\mu}; b_{\lambda\mu}) : X_\mu \rightarrow X_\lambda\}$  shall be called a *projective system of bivariant-class-equipped morphisms*.

For a bivariant theory  $\mathbb{B}$  on the category  $\mathcal{C}$  and for a projective system  $\{(\pi_{\lambda\mu}; b_{\lambda\mu}) : X_\mu \rightarrow X_\lambda\}$  of bivariant-class-equipped morphisms, the inductive limit

$$\varinjlim_{\Lambda} \left\{ B_*(X_\lambda), b_{\lambda\mu} \bullet : B_*(X_\lambda) \rightarrow B_*(X_\mu) \right\}$$

shall be denoted by

$$B_*^{\text{pro}}(X_\infty; \{b_{\lambda\mu}\})$$

emphasizing the projective system  $\{b_{\lambda\mu}\}$  of bivariant classes, because the above inductive limit surely depends on the choice of it. For examples, in the above theorems we have that

$$F^{\text{pro}}(X_\infty) = F_*^{\text{pro}}(X_\infty; \{\mathbb{1}_{\pi_{\lambda\mu}}\})$$

$$\mathbf{K}_0^{\text{pro}}(X_\infty) = K_*^{\text{pro}}(X_\infty; \{[\pi_{\lambda\mu}]\}).$$

Theorem (3.5)(i) is generalized to the following theorem:

**Theorem (3.13).** *Let  $\{(\pi_{n(n+1)}, \alpha_{n(n+1)}) : X_{n+1} \rightarrow X_n\}$  be a projective system of bivariant-class-equipped morphisms of topologically connected algebraic varieties with  $\alpha_{n(n+1)} \in \mathbb{F}(X_{n+1} \rightarrow X_n)$ . And assume that the (of course constant) Euler–Poincaré characteristic  $\chi(\pi_{n(n+1)}^{-1}(y); \alpha_{n(n+1)})$  of  $\alpha_{n(n+1)}$  restricted to each fiber  $\pi_{n(n+1)}^{-1}(y)$  is non-zero and it shall be denoted by  $\chi_f(\alpha_{n(n+1)})$ . And we set  $\chi_f(\alpha_{01}) := 1$ . Then the canonical Euler–Poincaré (pro)characteristic homomorphism*

$$\chi^{\text{pro}} : F^{\text{pro}}(X_\infty; \{\alpha_{n(n+1)}\}) \rightarrow \mathbb{Q}$$

is described by

$$\chi^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\chi(\alpha_n)}{\chi_f(\alpha_{01}) \cdot \chi_f(\alpha_{12}) \cdots \chi_f(\alpha_{(n-1)n})}.$$

*Proof.* Let  $(f, \alpha) : X \rightarrow Y$  be a bivariant-class-equipped morphism of topologically connected algebraic varieties with  $\alpha \in \mathbb{F}(X \xrightarrow{f} Y)$ . It follows from Proposition (3.10) that the Euler–Poincaré characteristic  $\chi(f^{-1}(y); \alpha)$  of  $\alpha$  restricted to each fiber  $f^{-1}(y)$  is constant (and non-zero by assumption). So, if it is denoted by  $\chi_f(\alpha)$ , then  $f_*\alpha = \chi_f(\alpha) \cdot \mathbb{1}_Y$ . Then to prove the theorem it suffices to see that we have the following commutative diagram:

$$\begin{array}{ccc} F(Y) & \xrightarrow{\alpha \bullet} & F(X) \\ \chi \downarrow & & \downarrow \chi \\ \mathbb{Z} & \xrightarrow{\times \chi_f(\alpha)} & \mathbb{Z}. \end{array}$$

To see this, we need the *projection formula* that for a morphism  $f : X \rightarrow Y$  and constructible functions  $\alpha \in F(X)$  and  $\beta \in F(Y)$

$$f_*(\alpha \cdot f^*\beta) = (f_*\alpha) \cdot \beta.$$

Then, using this projection formula we have

$$\begin{aligned} \chi(\alpha \bullet \beta) &= \chi(\alpha \cdot f^*\beta) \\ &= \chi(f_*\alpha \cdot \beta) \\ &= \chi((\chi_f(\alpha) \cdot \mathbb{1}_Y) \cdot \beta) \\ &= \chi_f(\alpha) \cdot \chi(\beta) \end{aligned}$$

Thus we get the above commutative diagram.  $\square$

Theorem (3.10), Theorem (3.11) and Theorem (3.12) are generalized to the following theorem:

**Theorem (3.14).** (i) Let  $\gamma : \mathbb{B} \rightarrow \mathbb{B}'$  be a Grothendieck transformation between two bivariant theories  $\mathbb{B}, \mathbb{B}' : \mathcal{C} \rightarrow \mathcal{C}'$  and let  $\{(\pi_{\lambda\mu}; b_{\lambda\mu}) : X_\mu \rightarrow X_\lambda\}$  be a projective system of bivariant-class-equipped morphisms. Then we get the following pro-version of the natural transformation  $\gamma_* : B_* \rightarrow B'_*$ :

$$\gamma_*^{\text{pro}} : B_*^{\text{pro}}(X_\infty; \{b_{\lambda\mu}\}) \rightarrow B'_*{}^{\text{pro}}(X_\infty; \{\gamma(b_{\lambda\mu})\}).$$

(ii) Let  $\{f_\lambda : Y_{\xi(\lambda)} \rightarrow X_\lambda\}$  be a fiber-square pro-morphism between two projective systems of bivariant-class-equipped morphisms such that  $b_{\xi(\lambda)\xi(\mu)} = f_\lambda^\star b_{\lambda\mu}$ . Then we have the following commutative diagram:

$$\begin{array}{ccc} B_*^{\text{pro}}(Y_\infty) & \xrightarrow{\gamma_*^{\text{pro}}} & B'_*{}^{\text{pro}}(Y_\infty) \\ f_{\infty*} \downarrow & & \downarrow f_{\infty*} \\ B_*^{\text{pro}}(X_\infty) & \xrightarrow{\gamma_*^{\text{pro}}} & B'_*{}^{\text{pro}}(X_\infty). \end{array}$$

$\square$

We hope to be able to do further investigations on characteristic classes (in particular, characteristic classes having a bivariant version) of proalgebraic varieties and some applications of them.

§4  $\chi$ -STABLE PROCONSTRUCTIBLE FUNCTIONS

In the previous section we have dealt with the case when the Euler–Poincaré characteristic of the fibers of each structure (or bonding) morphism, or the Euler–Poincaré characteristic of a constructible function restricted to the fibers of each structure morphism is non-zero constant. In this section we address ourselves to more general cases when this condition does not necessarily hold.

Let  $\{p_{\lambda\mu}\}$  be a system of non-zero integers indexed by the directed set  $\Lambda$ . If the following holds, then  $\{p_{\lambda\mu}\}$  shall be called a projective system:

$$p_{\lambda\lambda} = 1 \quad \text{and} \quad p_{\lambda\mu} \cdot p_{\mu\nu} = p_{\lambda\nu} \quad (\lambda < \mu < \nu).$$

For each  $\lambda \in \Lambda$  we define the following subgroup of  $F(X_\lambda)$ :

$$F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\lambda) := \left\{ \alpha_\lambda \in F(X_\lambda) \mid \chi(\pi_{\lambda\mu}^* \alpha_\lambda) = p_{\lambda\mu} \cdot \chi(\alpha_\lambda) \text{ for any } \mu > \lambda \right\}.$$

For each  $\lambda \in \Lambda$ , an element of  $F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\lambda)$  is called a  $\chi$ -stable constructible function with respect to the projective system  $\{p_{\lambda\mu}\}$  of non-zero integers. Then it is easy to see that for each structure morphism  $\pi_{\lambda\mu} : X_\mu \rightarrow X_\lambda$  the pullback homomorphism  $\pi_{\lambda\mu}^* : F(X_\lambda) \rightarrow F(X_\mu)$  preserves  $\chi$ -stable constructible functions with respect to the projective system  $\{p_{\lambda\mu}\}$  of non-zero integers, namely it induces the homomorphism (using the same symbol):

$$\pi_{\lambda\mu}^* : F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\lambda) \rightarrow F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\mu)$$

which implies that we get the inductive system

$$\left\{ F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\lambda), \pi_{\lambda\mu}^* : F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\lambda) \rightarrow F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\mu) \quad (\lambda < \mu) \right\}.$$

Then for a proalgebraic variety  $X_\infty = \varinjlim_{\lambda \in \Lambda} X_\lambda$  we consider the inductive limit of the above inductive system and it shall be denoted by

$$F_{\{p_{\lambda\mu}\}}^{\text{st.pro}}(X_\infty)$$

and an element of this group shall be called a  $\chi$ -stable proconstructible function on the proalgebraic variety  $X_\infty$  with respect to the projective system  $\{p_{\lambda\mu}\}$  of non-zero integers. We can see that this subgroup can be also directly defined as follows:

$$\left\{ [\alpha_\lambda] \in F^{\text{pro}}(X_\infty) \mid \chi(\pi_{\lambda\mu}^* \alpha_\lambda) = p_{\lambda\mu} \cdot \chi(\alpha_\lambda) \quad (\lambda < \mu) \right\}.$$

For each structure morphism  $\pi_{\lambda\mu} : X_\mu \rightarrow X_\lambda$  we get the following commutative diagram

$$\begin{array}{ccc} F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\lambda) & \xrightarrow{\pi_{\lambda\mu}} & F_{\{p_{\lambda\mu}\}}^{\text{st}}(X_\mu) \\ \chi \downarrow & & \downarrow \chi \\ \mathbb{Z} & \xrightarrow{\times p_{\lambda\mu}} & \mathbb{Z}. \end{array}$$

Thus we can get the following theorem:

**Theorem (4.1).** *For a proalgebraic variety  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$  and a projective system  $\{p_{\lambda\mu}\}$  of non-zero integers, we get the proalgebraic Euler–Poincaré characteristic homomorphism*

$$\chi_{\{p_{\lambda\mu}\}}^{\text{st.pro}} : F_{\{p_{\lambda\mu}\}}^{\text{st.pro}}(X_\infty) \rightarrow \varprojlim_{\lambda \in \Lambda} \left\{ \times p_{\lambda\mu} : \mathbb{Z} \rightarrow \mathbb{Z} \right\}.$$

In particular, when it comes to the case when  $\Lambda = \mathbb{N}$ , we get the following theorem

**Theorem (4.2).** *For a proalgebraic variety  $X_\infty = \varprojlim_{n \in \mathbb{N}} X_n$  and a projective system  $\{p_{nm}\}$  of non-zero integers, we have the following canonical proalgebraic Euler–Poincaré characteristic homomorphism*

$$\chi_{\{p_{nm}\}}^{\text{st.pro}} : F_{\{p_{nm}\}}^{\text{st.pro}}(X_\infty) \rightarrow \mathbb{Q}$$

which is defined by

$$\chi_{\{p_{nm}\}}^{\text{st.pro}} \left( \sum_n [\alpha_n] \right) := \sum_n \frac{\chi(\alpha_n)}{p_{01} \cdot p_{12} \cdot p_{23} \cdots p_{(n-1)n}}.$$

Here we set  $p_{01} := 1$ .

In the previous section all proconstructible functions are  $\chi$ -stable with respect to some projective systems of non-zero integers.

## §5 MOTIVIC MEASURES

The theorems or observations obtained concerning Chern–Schwartz–MacPherson classes or Euler–Poincaré characteristics of proalgebraic varieties in the previous section led the author to realize some deep connections to motivic measures and motivic integrations, in which the Grothendieck ring of complex proalgebraic varieties is a crucial ingredient.

First we recall the usual Grothendieck ring of algebraic varieties. Let  $\mathcal{V}_{\mathbb{C}}$  denote the category of complex algebraic varieties. Then the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$  of complex algebraic varieties is the free abelian group generated by the isomorphism classes of varieties modulo the subgroup generated by elements of the form  $[V] - [V'] - [V \setminus V']$  for a closed subset  $V' \subset V$  with the ring structure  $[V] \cdot [W] := [V \times W]$ . There are distinguished elements in  $K_0(\mathcal{V}_{\mathbb{C}})$ :  $\mathbb{1}$  is the class  $[p]$  of a point  $p$  and  $\mathbb{L}$  is the Tate class  $[\mathbb{C}]$  of the affine line  $\mathbb{C}$ . From this definition, we can see that any constructible set of a variety determines an element in the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$ . Provisionally the element  $[V]$  in the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$  is called the *Grothendieck “motivic” class of  $V$*  and let us denote it by  $\Gamma(V)$ . Hence we get the following homomorphism, called the *Grothendieck “motivic” class homomorphism*: for any variety  $X$

$$\Gamma : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}}),$$

which is defined by

$$\Gamma(\alpha) = \sum_{n \in \mathbb{Z}} n [\alpha^{-1}(n)].$$



Or  $\Gamma(\sum a_V \mathbb{1}_V) := \sum a_V [V]$  where  $V$  is a constructible set in  $X$  and  $a_V \in \mathbb{Z}$ .

This Grothendieck “motivic” class homomorphism is *tautological* and its more “geometric” one is the Euler–Poincaré characteristic homomorphism  $\chi : F(X) \rightarrow \mathbb{Z}$ . In the previous section we have generalized  $\chi : F(X) \rightarrow \mathbb{Z}$  to the category of proalgebraic varieties. Thus our problem is to generalize the Grothendieck “motivic” class homomorphism  $\Gamma : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})$  to the category of proalgebraic varieties.

**Problem (5.1).** *Let  $X_{\infty} = \varprojlim_{\lambda \in \Lambda} X_{\lambda}$  be a proalgebraic variety. Then describe a proalgebraic Grothendieck “motivic” (pro)class homomorphism*

$$\Gamma^{\text{pro}} : F(X_{\infty}) \rightarrow \sharp$$

*with some reasonable algebraic object  $\sharp$ .*

It seems that one cannot expect a general formula. However, we can get a Grothendieck “motivic” class homomorphism version of Theorem (3.5), if we consider the projective limit of a projective system of Zariski (not in the usual topology) locally trivial fiber bundles. Indeed, the Grothendieck “motivic” (pro)class homomorphism version of (3.5.2) is the following:

$$(5.2) \quad \begin{array}{ccc} F(X_n) & \xrightarrow{\pi_{(n-1)n}^*} & F(X_{n+1}) \\ \Gamma \downarrow & & \downarrow \Gamma \\ K_0(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\times [F_n]} & K_0(\mathcal{V}_{\mathbb{C}}). \end{array}$$

However, when we consider a localization of the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$ , we need to be a bit careful. Namely, the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$  is not a domain unlike the ring  $\mathbb{Z}$  of integers. Indeed, here is a very recent result due to B. Poonen [Po, Theorem 1]:

**Theorem (5.3).** *(B. Poonen) Suppose that  $k$  is a field of characteristic zero. Then the Grothendieck ring  $K_0(\mathcal{V}_k)$  of  $k$ -varieties is not a domain.*

Recall that for any ring  $R$  and  $S$  the multiplicatively closed set of all non-zero divisors of  $R$ , the quotient ring  $R_S$  of  $R$  with denominator set being  $S$  is called the *total (or full) quotient ring of  $R$*  and denoted by  $Q(R)$  (eg. see [Kunz]). So, let  $Q(K_0(\mathcal{V}_{\mathbb{C}}))$  denote the total quotient ring of the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$ . Then, very simply thinking, the “motivic” proclass version of the canonical Euler–Poincaré (pro)characteristic homomorphism  $\chi^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow \mathbb{Q}$  given in Theorem (3.5) could be the homomorphism

$$\Gamma^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow Q(K_0(\mathcal{V}_{\mathbb{C}}))$$

defined by

$$\Gamma^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\Gamma(\alpha_n)}{[F_0][F_1][F_2] \cdots [F_{n-1}]}.$$

Here  $F_0$  is defined to be a point, i.e.,  $[F_0] := \mathbb{1}$ , and also we assume that each  $[F_j]$  is not a zero-divisor.

One serious problem of this definition is that it is in general hard to check whether the class  $[F_j]$  is a non-zero divisor or not; indeed, we do not know whether even the class  $[\mathbb{P}^j]$  of the projective space is a non-zero divisor (which Willem Veys pointed out to the author). So, in order to avoid this problem, we consider the following simple localization: let  $\mathcal{F}$  be the multiplicative set consisting of all the finite products of  $[F_j]^{m_j}$ , i.e.,

$$\mathcal{F} := \left\{ [F_{j_1}]^{m_1} [F_{j_2}]^{m_2} \cdots [F_{j_s}]^{m_s} \mid j_i \in \mathbb{N}, m_i \in \mathbb{N} \right\}.$$

Then the quotient ring  $\mathcal{F}^{-1}(K_0(\mathcal{V}_{\mathbb{C}}))$  shall be denoted by  $K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$ . With this definition we get the following theorem:

**Theorem (5.4).** *Let  $X_{\infty} = \varprojlim_n X_n$  be a proalgebraic variety such that each structure morphism  $\pi_{n(n+1)} : X_{n+1} \rightarrow X_n$  satisfies the condition that  $[X_{n+1}] = [X_n][F_n]$  for some variety  $F_n$  for each  $n$ , for example,  $\pi_{n,n+1} : X_{n+1} \rightarrow X_n$  is a Zariski locally trivial fiber bundle with fiber variety being  $F_n$ .*

(i) *The canonical Grothendieck “motivic” proclass homomorphism,*

$$\Gamma^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$$

*is described by*

$$\Gamma^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\Gamma(\alpha_n)}{[F_0][F_1][F_2] \cdots [F_{n-1}]}.$$

*Here  $F_0$  is defined to be a point, i.e.,  $[F_0] := \mathbb{1}$ .*

(ii) *In particular, if all the fibers are the same, say  $F_n = W$  for any  $n$ , then the canonical Grothendieck “motivic” (pro)class homomorphism*

$$\Gamma^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$$

*is described by*

$$\Gamma^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\Gamma(\alpha_n)}{[W]^{n-1}}.$$

*In this special case the quotient ring  $K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$  shall be simply denoted by  $K_0(\mathcal{V}_{\mathbb{C}})_{[W]}$ . And another non-canonical one  $\Gamma_w^{\text{pro}}$  is*

$$\Gamma_w^{\text{pro}} \left( \sum_n [\alpha_n] \right) = \sum_n \frac{\Gamma(\alpha_n)}{[W]^{n-1+w}}.$$

□

Let  $W$  be a variety and let  $X \widetilde{\times} W^k$  be a fiber bundle inductively defined as a fiber bundle over  $X \widetilde{\times} W^{k-1}$  with fiber being  $W$ . Then the projective limit  $\varprojlim_k X \widetilde{\times} W^k$  shall

be denoted by  $X \widetilde{\times} W^{\mathbb{N}}$ . Then, for the characteristic function  $\mathbb{1}_{X \widetilde{\times} W^{\mathbb{N}}} = [\mathbb{1}_{X \widetilde{\times} W^{k-1}}]$  for any  $k$ , we have

$$\Gamma^{\text{pro}}(\mathbb{1}_{X \widetilde{\times} W^{\mathbb{N}}}) = \frac{[X] \cdot [W]^{k-1}}{[W]^{k-1}} = [X],$$

which is called *the Grothendieck “motivic” proclass of the proalgebraic variety  $X \widetilde{\times} W^{\mathbb{N}}$*  and denoted by  $[X \widetilde{\times} W^{\mathbb{N}}]^{\text{pro}}$  and sometimes called a “fiber bundle” over an algebraic variety  $X$  with fiber being the proalgebraic variety  $W^{\mathbb{N}}$ , abusing words.

As to the Grothendieck class “motivic” homomorphism version of Theorem (3.5)(iii), we do not know whether it is true or not. Thus we want to ask the following question:

**Question (5.5).** *Is the canonical Grothendieck “motivic” (pro)class homomorphism*

$$\Gamma^{\text{pro}} : F^{\text{pro}}(X_{\infty}) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$$

*is surjective if each  $F_n = \mathbb{P}^n$  the projective space of dimension  $n$  ?*

Since the Grothendieck “motivic” (pro)classes of proconstructible functions on the above proalgebraic variety  $X \widetilde{\times} W^{\mathbb{N}}$  are in the localization  $K_0(\mathcal{V}_{\mathbb{C}})_{[W]}$  of the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$ , our very naïve question is:

**Question (5.6).** *What is the Grothendieck ring of the category  $\mathcal{V}_{\mathbb{C}}^{\text{pro}}$  of proalgebraic varieties ? What is  $K_0(\mathcal{V}_{\mathbb{C}}^{\text{pro}})$  ?*

At the moment we do not have an answer for this question. However, we can give a plausible answer for this in a special category of proalgebraic varieties, which are the above “fiber bundles” over complex algebraic varieties with fiber being the proalgebraic variety  $W^{\mathbb{N}}$ . Infinite “fiberings with fiber  $W$ ” give rise to the following inductive system

$$\dots \xrightarrow{\times[W]} K_0(\mathcal{V}_{\mathbb{C}}) \xrightarrow{\times[W]} K_0(\mathcal{V}_{\mathbb{C}}) \xrightarrow{\times[W]} K_0(\mathcal{V}_{\mathbb{C}}) \xrightarrow{\times[W]} \dots$$

the colimit of which would be the localization  $K_0(\mathcal{V}_{\mathbb{C}})_{[W]}$  of the Grothendieck ring  $K_0(\mathcal{V}_{\mathbb{C}})$ .

Now we discuss motivic measures of the arc space  $\mathcal{L}(X)$  of an algebraic variety  $X$ . The arc space  $\mathcal{L}(X)$  of an algebraic variety  $X$  is defined to be the projective limit of truncated arc varieties  $\mathcal{L}_n(X)$  and projection  $\pi_{n-1,n} : \mathcal{L}_n(X) \rightarrow \mathcal{L}_{n-1}(X)$ . Thus the arc space  $\mathcal{L}(X)$  is a proalgebraic variety. Let  $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$  be the canonical projection or the  $n$ -th truncated morphism. Then a subset  $A$  of the arc space  $\mathcal{L}(X)$  is called a *cylinder set* [Cr] or simply a *constructible set* [DL 1, DL 2] if  $A = \pi_n^{-1}(C_n)$  for a constructible set  $C_n$  in the  $n$ -th arc space  $\mathcal{L}_n(X)$  for some integer  $n \geq 0$ . In this paper, to avoid some possible confusion, we take the name “cylinder set”.

A  $\mathbb{Z}$ -valued function  $\alpha : \mathcal{L}(X) \rightarrow \mathbb{Z}$  on the arc space  $\mathcal{L}(X)$  is called a *cylinder function* if  $\alpha$  is a linear combination of characteristic functions on cylinder sets. It turns out that the abelian group of cylinder functions on the arc space  $\mathcal{L}(X)$ , denoted by  $F^{\text{cyl}}(\mathcal{L}(X))$ , is isomorphic to the abelian group  $F^{\text{pro}}(\mathcal{L}(X))$ , as shown below.

Our previous definition of a proconstructible function in  $F^{\text{pro}}(X_{\infty}) = \varinjlim_{\lambda \in \Lambda} F(X_{\lambda})$  as a function on  $X_{\infty}$  means that we define the “functionization” homomorphism

$$\Psi : \varinjlim_{\lambda \in \Lambda} F(X_{\lambda}) \rightarrow \text{Fun}(X_{\infty}, \mathbb{Z}) \quad \text{defined by} \quad \Psi([\alpha_{\mu}])((x_{\lambda})) := \alpha_{\mu}(x_{\mu}).$$

Here  $Fun(X_\infty, \mathbb{Z})$  is the abelian group consisting of functions or mappings from  $X_\infty$  to  $\mathbb{Z}$ . Of course, the target  $\mathbb{Z}$  can be replaced by any other ring containing the integers  $\mathbb{Z}$ .

One can describe the above in a much fancier way as follows. Let  $\pi_\lambda : X_\infty \rightarrow X_\lambda$  denote the canonical projection induced from the projection  $\prod_\lambda X_\lambda \rightarrow X_\lambda$ . Consider the following commutative diagram for each  $\lambda \in \Lambda$ :

$$\begin{array}{ccc}
 F(X_\lambda) & & \\
 \downarrow \pi_{\lambda\mu}^* & \searrow \pi_\lambda^* & \\
 & & Fun(X_\infty, \mathbb{Z}) \\
 & \nearrow \pi_\mu^* & \\
 F(X_\mu) & & 
 \end{array}$$

Then it follows as a standard fact in the theory of inductive limits that the “functionization” homomorphism  $\Psi : \varinjlim_{\lambda \in \Lambda} F(X_\lambda) \rightarrow Fun(X_\infty, \mathbb{Z})$  is the unique homomorphism such that the following diagram commutes:

$$\begin{array}{ccccc}
 & F(X_\lambda) & & & \\
 \swarrow \rho^\lambda & & \searrow \pi_\lambda^* & & \\
 \varinjlim_{\lambda \in \Lambda} F(X_\lambda) & \xrightarrow{\Psi} & Fun(X_\infty, \mathbb{Z}) & & 
 \end{array}$$

To avoid some possible confusion, the image  $\Psi([\alpha_\lambda]) = \pi_\lambda^* \alpha_\lambda$  shall be denoted by  $[\alpha_\lambda]_\infty$ . For a constructible set  $W_\lambda \in X_\lambda$ ,  $[\mathbb{1}_{W_\lambda}]_\infty$  shall be also called a *procharacteristic function*. Then Image  $\Psi$  is generated by all the procharacteristic functions  $[\mathbb{1}_{W_\lambda}]_\infty$  with  $W_\lambda \subset X_\lambda$  constructible sets. By the definition we have

$$[\mathbb{1}_{W_\lambda}]_\infty = \mathbb{1}_{\pi_\lambda^{-1}(W_\lambda)}.$$

Thus, mimicking the term of cylinder set [Cr], the set  $\pi_\lambda^{-1}(W_\lambda) \subset X_\infty$  for a constructible set  $W_\lambda \in X_\lambda$  is also called a  $\lambda$ -cylinder set or a cylinder set of level  $\lambda$ . Thus  $[\mathbb{1}_{W_\lambda}]_\infty$  is a characteristic function on the cylinder set, so a cylinder function. It is easy to see the following

**Proposition (5.7).** *For  $\lambda < \mu$  and two constructible sets  $W_\lambda \in X_\lambda$  and  $W_\mu \in X_\mu$  we have*

$$\pi_\lambda^{-1}(W_\lambda) = \pi_\mu^{-1}(W_\mu) \iff W_\mu = \pi_{\lambda\mu}^{-1}(W_\lambda) \iff \mathbb{1}_{W_\mu} = \pi_{\lambda\mu}^* (\mathbb{1}_{W_\lambda}).$$

Thus we can see that the notions of procharacteristic function and of cylinder set are *equivalent*.

Thus we define

**Definition (5.8).** For a proalgebraic variety  $X_\infty = \varinjlim_{\lambda \in \Lambda} X_\lambda$

$$F^{\text{cyl}}(X_\infty) := \text{Image} \left( \Psi : \varinjlim_{\lambda \in \Lambda} F(X_\lambda) \rightarrow \text{Fun}(X_\infty, \mathbb{Z}) \right) = \bigcup_{\mu} \pi_\mu^*(F(X_\mu)).$$

And an element of  $F^{\text{cyl}}(X_\infty)$  is called a cylinder function on  $X_\infty$ .

*Remark (5.9).* As long as the support is concerned,

$$\text{supp}[\mathbb{1}_{W_\lambda}] = \text{supp}[\mathbb{1}_{W_\lambda}]_\infty = \pi_\lambda^{-1}(W_\lambda),$$

whether the characteristic function  $\mathbb{1}_{W_\lambda} \in F(X_\lambda)$  is considered as the proconstructible function  $[\mathbb{1}_{W_\lambda}] \in F^{\text{pro}}(X_\infty)$  or considered as the cylinder function  $[\mathbb{1}_{W_\lambda}]_\infty \in F^{\text{cyl}}(X_\infty)$ . Note that our results obtained so far of course hold with nothing changed at all even if  $F^{\text{pro}}(X_\infty)$  is replaced by  $F^{\text{cyl}}(X_\infty)$ .

In an earlier version of the present paper, we considered only  $F^{\text{pro}}(X_\infty)$ . However, in a discussion with Jörg Schürmann, he suggested to also consider the image of the above “functionization” homomorphism, pointing out that in general the inductive limit would have more informations than the image, i.e., that  $\Psi$  may not be necessarily injective.

In some cases it is injective. For example, if all the homomorphism  $\pi_\lambda^* : F(X_\lambda) \rightarrow \text{Fun}(X_\infty, \mathbb{Z})$  are injective, which is in turn equivalent to the condition that all the projections  $\pi_\lambda : X_\infty \rightarrow X_\lambda$  is surjective, then the “functionization” homomorphism  $\Psi : \varinjlim_{\lambda \in \Lambda} F(X_\lambda) \rightarrow \text{Fun}(X_\infty, \mathbb{Z})$  is also injective since the inductive limit is an exact functor. So, we get the following lemma:

**Proposition (5.10).** *If all the structure morphisms  $\pi_{\mu\lambda} : X_\mu \rightarrow X_\lambda$  (for  $\mu < \lambda$ ) are surjective, then the “functionization” homomorphism  $\Psi : \varinjlim_{\lambda \in \Lambda} F(X_\lambda) \rightarrow \text{Fun}(X_\infty, \mathbb{Z})$  is injective.*

In the case of the arc space  $\mathcal{L}(X)$ , since each structure morphism  $\pi_{n,n+1} : \mathcal{L}_{n+1}(X) \rightarrow \mathcal{L}_n(X)$  is always surjective, we get the following

**Corollary (5.11).** *For the arc space  $\mathcal{L}(X)$  we have the canonical isomorphism*

$$F^{\text{pro}}(\mathcal{L}(X)) \cong F^{\text{cyl}}(\mathcal{L}(X)).$$

Suppose that  $\Psi([\alpha_\mu]) = 0$ , which means that  $\Psi([\alpha_\mu])((x_\lambda) = \alpha_\mu(x_\mu) = 0$  for any  $(x_\lambda) \in X_\infty$ . Hence we have

$$\alpha_\mu(\pi_\mu(X_\infty)) = 0.$$

At the moment we do not know whether we can conclude  $[\alpha_\mu] = 0$  from this condition. There is a very simple example such that  $\alpha_\mu(\pi_\mu(X_\infty)) = 0$ ,  $\pi_\mu(X_\infty) \neq X_\mu$  and  $\alpha_\mu \neq 0$ , but  $[\alpha_\mu] = 0$ : Let  $X_1 = \{a, b\}$  be a space of two different points, and let  $X_n = \{a\}$  for any  $n > 1$ . Let  $\pi_{1,2} : X_2 \rightarrow X_1$  is the injection map sending  $a$  to  $a$  and the other structure morphism  $\pi_{n(n+1)} : X_{n+1} \rightarrow X_n$  is the identity for  $n > 1$ . Then the projective limit  $X_\infty = \{(a)\}$  consists of one point  $(a, a, a, \dots)$ . Let  $\alpha_1 = p \cdot \mathbb{1}_b \in F(X_1)$ . Then we have  $\alpha_1(\pi_1(X_\infty)) = 0$ ,  $\pi_1(X_\infty) \neq X_1$  and  $\alpha_1 \neq 0$ , but  $[\alpha_1] = 0$ . We suspect that in general the “functionization” homomorphism  $\Psi$  is not necessarily injective, but we have not been able to find such an example yet.

**Corollary (5.12).** *When  $X$  is a nonsingular variety of dimension  $d$ , we have the following canonical Grothendieck “motivic” (pro)class homomorphism*

$$\Gamma^{\text{pro}} : F^{\text{cyl}}(\mathcal{L}(X)) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})_{[\mathbb{L}^d]}$$

*is described by*

$$\Gamma^{\text{pro}} \left( \sum_n [\alpha_n]_{\infty} \right) = \sum_n \frac{\Gamma(\alpha_n)}{[\mathbb{L}]^{nd}}$$

*and another non-canonical one  $\Gamma_w^{\text{pro}}$  is*

$$\Gamma_w^{\text{pro}} \left( \sum_n [\alpha_n]_{\infty} \right) = \sum_n \frac{\Gamma(\alpha_n)}{[\mathbb{L}]^{(n+w)d}}$$

*In particular, we get that  $[\mathcal{L}(X)]^{\text{pro}} = [\mathbb{1}_{\mathcal{L}(X)}]^{\text{pro}} = [X]$ .  $\square$*

Note that in the case of arc space  $\mathcal{L}(X)$ , since  $\mathcal{L}_0(X) = X$ , the index set is not  $\mathbb{N}$  but  $\{0\} \cup \mathbb{N}$ . Hence the canonical one is not  $\Gamma^{\text{pro}} \left( \sum_n [\alpha_n]_{\infty} \right) = \sum_n \frac{\Gamma(\alpha_n)}{[\mathbb{L}]^{(n-1)d}}$ .

Therefore we can see that our proalgebraic Grothendieck “motivic” (pro)class homomorphism  $\Gamma : F^{\text{pro}}(X_{\infty}) \rightarrow Q(K_0(\mathcal{V}_{\mathbb{C}}))$  is a generalization of the so-called motivic measure in the case when the base variety  $X$  is smooth.

Thus the theorem and corollary imply that in the theory of motivic measures and motivic integrations the notion of cylinder set is a right one from the proalgebraic viewpoint or from the viewpoint of pro-category.

If  $X$  is singular, the arc space  $\mathcal{L}(X)$  is *not* the projective limit of a projective system of Zariski locally trivial fiber bundles with fiber being  $\mathbb{C}^{\dim X}$  any longer and each projection morphism  $\pi_{(n-1)n} : \mathcal{L}_n(X) \rightarrow \mathcal{L}_{n-1}(X)$  is complicated and thus as a proalgebraic variety  $\mathcal{L}(X)$  is complicated and in general we do not know whether there exists a well-defined homomorphism  $\Gamma : F^{\text{pro}}(\mathcal{L}(X)) \rightarrow \text{Quot}(K_0(\mathcal{V}_{\mathbb{C}}))$  for some suitable quotient ring  $\text{Quot}(K_0(\mathcal{V}_{\mathbb{C}}))$ , also we do not know whether there exists a well-defined proalgebraic Euler–Poincaré characteristic homomorphism  $\chi^{\text{pro}} : F^{\text{pro}}(\mathcal{L}(X)) \rightarrow \mathbb{Q}$ .

A crucial ingredient in studying motivic measure or motivic integration is the so-called *stable set* of the arc space  $\mathcal{L}(X)$ .

**Definition (5.13).** A subset  $A$  of the arc space  $\mathcal{L}(X)$  is called a *stable set* if it is a cylinder set, i.e.,  $A = \pi_n^{-1}(C_n)$  for a constructible set  $C_n$  in the  $n$ -th arc space  $\mathcal{L}_n(X)$ , such that the restriction of each projection  $\pi_{m(m+1)}|_{\pi_{m+1}(A)} : \pi_{m+1}(A) \rightarrow \pi_m(A)$  for each  $m \geq n$  is a Zariski locally fiber bundle with the fiber being  $\mathbb{C}^{\dim X}$ , in other words, the restriction of the projection  $\pi_n : \mathcal{L}(X) \rightarrow \mathcal{L}_n(X)$  to the constructible set  $C_n$  is  $C_n \times (\mathbb{C}^{\dim X})^{\mathbb{N}}$ . And a *stable function* on the arc space  $\mathcal{L}(X)$  is a  $\mathbb{Z}$ -valued function constant along stable sets.

In fact, in a similar way as Definition (5.13) or as in §4, we can define a “*motivic*” *stable proconstructible function* on any proalgebraic variety  $X_{\infty}$  as follows.

Let  $\{[F_{\lambda\mu}]\}$  be a system of Grothendieck classes  $[F_{\lambda\mu}] \in K_0(\mathcal{V}_{\mathbb{C}})$  indexed by the directed set  $\Lambda$ . As in §4, if the following holds, then  $\{[F_{\lambda\mu}]\}$  shall be called a projective system:

$$[F_{\lambda\lambda}] = \mathbb{1} \quad \text{and} \quad [F_{\lambda\mu}] \cdot [F_{\mu\nu}] = [F_{\lambda\nu}] \quad (\lambda < \mu < \nu).$$

For each  $\lambda \in \Lambda$  we define the following subgroup of  $F(X_\lambda)$ :

$$F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\lambda) := \left\{ \alpha_\lambda \in F(X_\lambda) \mid \Gamma(\pi_{\lambda\mu}^* \alpha_\lambda) = [F_{\lambda\mu}] \cdot \Gamma(\alpha_\lambda) \text{ for any } \mu > \lambda \right\}.$$

For each  $\lambda \in \Lambda$ , an element of  $F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\lambda)$  is called a  $\Gamma$ -stable constructible function with respect to the projective system  $\{[F_{\lambda\mu}]\}$  of Grothendieck classes. Then it is easy to see that for each structure morphism  $\pi_{\lambda\mu} : X_\mu \rightarrow X_\lambda$  the pullback homomorphism  $\pi_{\lambda\mu}^* : F(X_\lambda) \rightarrow F(X_\mu)$  preserves  $\Gamma$ -stable constructible functions with respect to the projective system  $\{[F_{\lambda\mu}]\}$  of Grothendieck classes, namely it induces the homomorphism (using the same symbol):

$$\pi_{\lambda\mu}^* : F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\lambda) \rightarrow F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\mu)$$

which implies that we get the inductive system

$$\left\{ F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\lambda), \pi_{\lambda\mu}^* : F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\lambda) \rightarrow F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\mu) \quad (\lambda < \mu) \right\}.$$

Then for a proalgebraic variety  $X_\infty = \varinjlim_{\lambda \in \Lambda} X_\lambda$  we consider the inductive limit of the above inductive system and it is denoted by

$$F_{\{[F_{\lambda\mu}]\}}^{\text{st.pro}}(X_\infty)$$

and an element of this group shall be called a  $\Gamma$ -stable proconstructible function on the proalgebraic variety  $X_\infty$  with respect to the projective system  $\{[F_{\lambda\mu}]\}$  of Grothendieck classes. We can see that this subgroup can be also directly defined as follows:

$$\left\{ [\alpha_\lambda] \in F^{\text{pro}}(X_\infty) \mid \Gamma(\pi_{\lambda\mu}^* \alpha_\lambda) = [F_{\lambda\mu}] \cdot \Gamma(\alpha_\lambda) \quad (\lambda < \mu) \right\}.$$

For each structure morphism  $\pi_{\lambda\mu} : X_\mu \rightarrow X_\lambda$  we get the following commutative diagram

$$\begin{array}{ccc} F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\lambda) & \xrightarrow{\pi_{\lambda\mu}} & F_{\{[F_{\lambda\mu}]\}}^{\text{st}}(X_\mu) \\ \Gamma \downarrow & & \downarrow \Gamma \\ K_0(\mathcal{V}_{\mathbb{C}}) & \xrightarrow{\times [F_{\lambda\mu}]} & K_0(\mathcal{V}_{\mathbb{C}}). \end{array}$$

Thus we can get the following theorem:

**Theorem (5.14).** *For a proalgebraic variety  $X_\infty = \varprojlim_{\lambda \in \Lambda} X_\lambda$  and a projective system  $\{[F_{\lambda\mu}]\}$  of Grothendieck classes, we get the Grothendieck “motivic” class homomorphism*

$$\Gamma_{\{[F_{\lambda\mu}]\}}^{\text{st.pro}} : F_{\{[F_{\lambda\mu}]\}}^{\text{st.pro}}(X_\infty) \rightarrow \varprojlim_{\lambda \in \Lambda} \left\{ \times [F_{\lambda\mu}] : K_0(\mathcal{V}_{\mathbb{C}}) \rightarrow K_0(\mathcal{V}_{\mathbb{C}}) \right\}.$$

In particular we get the following theorem, which is a “motivic” version of Theorem (4.1):

**Theorem (5.15).** *For a proalgebraic variety  $X_\infty = \varprojlim_{n \in \mathbb{N}} X_n$  and a projective system  $\{[F_{n,m}]\}$  of Grothendieck classes, we have the following canonical Grothendieck “motivic” class homomorphism*

$$\Gamma_{\{[F_{nm}]\}}^{\text{st.pro}} : F_{\{[F_{nm}]\}}^{\text{st.pro}}(X_\infty) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$$

which is defined by

$$\Gamma_{\{[F_{nm}]\}}^{\text{st.pro}} \left( \sum_n [\alpha_n] \right) := \sum_n \frac{\Gamma(\alpha_n)}{[F_{01}][F_{12}][F_{23}] \cdots [F_{(n-1)n}]}.$$

Here we set  $[F_{01}] := \mathbb{1}$  and  $\mathcal{F}$  is the multiplicative set consisting of all the finite products of  $[F_{j(j+1)}]^{m_j}$  as in Theorem (5.4).

In the case of the arc space  $\mathcal{L}(X)$ , the projective system  $\{[F_{nm}]\}$  of Grothendieck classes is such that  $[F_{nm}] = [\mathbb{C}^{(m-n)d}]$ . Thus Theorem (5.14) is a generalization of motivic measure when we deal with the arc space  $\mathcal{L}(X)$  of a possibly singular variety  $X$ .

*Remark (5.16).* So far we have looked at the two components of  $c_* : F(X) \rightarrow H_*(X)$ . So it is natural to consider whether or not there is a motivic version of the whole  $c_*$ , say,

$$\mu c_* : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})$$

such that the two distinguished “components” of  $\mu c_*$  are

$$\chi : F(X) \rightarrow \mathbb{Z} \quad \text{and} \quad \Gamma : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}}).$$

A very naïve and very simple-minded guess is the following “construction”: Suppose that for a constructible function  $\alpha \in F(X)$ ,

$$c_*(\alpha) = \chi(\alpha) + \sum_i a_i [V_i] \in H_*(X)$$

where  $[V_i]$  is the homology class represented by a subvariety  $V_i$  and  $a_i \in \mathbb{Z}$ . Then the motivic class of the whole  $c_*(\alpha)$ , denoted by  $\mu c_*(\alpha)$ , could be defined by the following tautological one:

$$\mu c_*(\alpha) = \chi(\alpha) \mathbb{1} + \sum_i a_i [V_i] \in K_0(\mathcal{V}_{\mathbb{C}}).$$

However, this definition is not well defined. So it remains to see whether or there exists a motivic version  $\mu c_* : F(X) \rightarrow K_0(\mathcal{V}_{\mathbb{C}})$ . If we can obtain a reasonable motivic version  $\mu c_*$ , then we expect that we can also get a proalgebraic version of  $\mu c_*$ .



## §6 A FEW REMARKS ON INTEGRATIONS

In this section we discuss a proalgebraic version of integration with respect to the proalgebraic Euler–Poincaré characteristic  $\chi^{\text{pro}}$  or the Grothendieck “motivic” class  $\Gamma^{\text{pro}}$  in the case when the directed set  $\Lambda = \mathbb{N}$ .

As remarked in the Introduction, the usual Euler–Poincaré characteristic  $\chi : F(X) \rightarrow \mathbb{Z}$  is described by, putting emphasis on integration,

$$(6.1) \quad \chi(\alpha) = \int_X \alpha d\chi = \sum_n n\chi(\alpha^{-1}(n)).$$

and furthermore, for a function  $f : \mathbb{Z} \rightarrow \mathbb{Z}$

$$(6.2) \quad \int_X f(\alpha) d\chi = \sum_n f(n)\chi(\alpha^{-1}(n)).$$

We remark that  $f(\alpha)$  should be more precisely expressed as the composite  $f \circ \alpha$ , but that we write it so for simplicity.

A very naïve proalgebraic version of (6.1) would be

$$\int_{X_\infty} \alpha d\chi^{\text{pro}} = \sum_n n\chi^{\text{pro}}(\alpha^{-1}(n))$$

for a proconstructible function  $\alpha \in F^{\text{pro}}(X_\infty)$ . Here comes out a problem. How do we define  $\chi^{\text{pro}}(\alpha^{-1}(n))$ ? So far,  $\chi^{\text{pro}}$  is defined on proconstructible functions or more explicitly defined on procharacteristic functions. As observed in Proposition (5.7), the procharacteristic function is equivalent to the cylinder set. Therefore  $\chi^{\text{pro}}(\alpha^{-1}(n))$  are defined on cylinder sets (cf. Remark (5.9)). However, we know that an arbitrary proconstructible function  $\alpha \in F^{\text{pro}}(X_\infty)$  does not necessarily satisfy the property that  $\alpha^{-1}(n)$  is always a cylinder set for any  $n$ . For example, consider the following situation: Let  $\{X_n, \pi_{n(n+1)} : X_{n+1} \rightarrow X_n\}$  be a projective system of algebraic varieties with each structure morphism  $\pi_{n(n+1)} : X_{n+1} \rightarrow X_n$  being surjective. Let  $W_1 \in X_1$  and  $W_2 \in X_2$  be constructible sets and consider the proconstructible function  $\alpha = [\mathbb{1}_{W_1}] + [\mathbb{1}_{W_2}]$ . Then we have

$$\alpha^{-1}(n) = \begin{cases} \pi_1^{-1}(W_1) \ominus \pi_2^{-1}(W_2) & \text{if } n = 1 \\ \pi_1^{-1}(W_1) \cap \pi_2^{-1}(W_2) & \text{if } n = 2 \\ \emptyset & \text{otherwise} \end{cases}$$

Here the symbol  $\ominus$  is the symmetric difference, i.e.,  $A \ominus B = (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$ .

If  $\pi_1^{-1}(W_1) \cap \pi_2^{-1}(W_2) = \emptyset$ , then  $\pi_1^{-1}(W_1) \ominus \pi_2^{-1}(W_2)$  is the disjoint union of two cylinder sets. Thus we can define  $\chi^{\text{pro}}(\alpha^{-1}(1))$ . However, if  $\pi_1^{-1}(W_1) \cap \pi_2^{-1}(W_2) \neq \emptyset$ , in general  $\pi_1^{-1}(W_1) \ominus \pi_2^{-1}(W_2)$  and  $\pi_1^{-1}(W_1) \cap \pi_2^{-1}(W_2)$  are not necessarily cylinder sets. Hence, to define the integration with respect to  $\chi^{\text{pro}}$  or  $\Gamma^{\text{pro}}$ , we need to restrict ourselves to proconstructible functions whose fibers are always cylinder sets.

For a proalgebraic variety  $X_\infty = \varprojlim_{n \in \mathbb{N}} X_n$  and a projective system  $\{p_{nm}\}$  of non-zero integers, consider a  $\chi$ -stable proconstructible function  $\alpha$  whose fibers are all cylinder sets and a function  $f : \mathbb{Z} \rightarrow \mathbb{Q}$ . Then the following integration is well-defined:

$$\int_{X_\infty} f(\alpha) d\chi_{\{p_{nm}\}}^{\text{st.pro}} := \sum_k f(k) \chi_{\{p_{nm}\}}^{\text{st.pro}}(\alpha^{-1}(k)).$$

Similarly, for a proalgebraic variety  $X_\infty = \varprojlim_{n \in \mathbb{N}} X_n$  and a projective system  $\{[F_{nm}]\}$  of Grothendieck classes, consider a  $\Gamma$ -stable proconstructible function  $\alpha$  whose fibers are all cylinder sets and a function  $f : \mathbb{Z} \rightarrow K_0(\mathcal{V}_{\mathbb{C}})_{\mathcal{F}}$ . Then the following “motivic” integration is well-defined:

$$\int_{X_\infty} f(\alpha) d\Gamma_{\{[F_{nm}]\}}^{\text{st.pro}} := \sum_k f(k) \Gamma_{\{[F_{nm}]\}}^{\text{st.pro}}(\alpha^{-1}(k)).$$

## §7 PRORESOLUTIONS

Finally we consider the following projective system of resolution of singularities: Let  $Y$  be a possibly singular variety and let  $\mathcal{RES}_Y$  be the collection  $\{(Y', g)\}$  of resolution of singularities  $g : Y' \rightarrow Y$  of  $Y$ , where  $Y'$  is nonsingular and  $g|_{Y' \setminus g^{-1}(Y_{\text{sing}})} : Y' \setminus g^{-1}(Y_{\text{sing}}) \rightarrow Y \setminus Y_{\text{sing}}$  is an isomorphism with  $Y_{\text{sing}}$  denoting the singular set of  $Y$ . When  $Y$  is nonsingular,  $\mathcal{RES}_Y$  is defined to be just  $\{(Y, \text{id}_Y)\}$ . The reason for this requirement is that otherwise  $\mathcal{RES}_Y$  consists of all automorphisms of  $Y$ , which is not necessary for our purpose.

For two elements  $g_1 : Y_1 \rightarrow Y$ ,  $g_2 : Y_2 \rightarrow Y$  of  $\mathcal{RES}_Y$  we define the order  $\leq$  as follows:  $g_1 \leq g_2$  if and only if there exists a morphism  $g_{12} : Y_2 \rightarrow Y_1$  such that  $g_2 = g_1 \circ g_{12}$ :

$$\begin{array}{ccc} Y_2 & \xrightarrow{g_{12}} & Y_1 \\ & \searrow g_2 & \swarrow g_1 \\ & Y & \end{array}$$

**Proposition (7.1).** *For a possibly singular variety  $Y$ , the ordered set  $(\mathcal{RES}_Y, \leq)$  is a directed set.*

*Proof.* Let  $g_1, g_2 \in \mathcal{RES}_Y$ . We want to show that there exists a desingularization  $g_3 : Y_3 \rightarrow Y$  such that  $g_1 \leq g_3$  and  $g_2 \leq g_3$ . Consider the fiber product

$$\begin{array}{ccc} Y_1 \times_Y Y_2 & \xrightarrow{\tilde{g}_1} & Y_2 \\ \tilde{g}_2 \downarrow & & \downarrow g_2 \\ Y_1 & \xrightarrow{g_1} & Y, \end{array}$$

and furthermore consider a resolution of singularities of  $Y_1 \times_Y Y_2$ :

$$\pi : \widetilde{Y_1 \times_Y Y_2} \rightarrow Y_1 \times_Y Y_2.$$

Let us set  $Y_3$  to be  $\widetilde{Y_1 \times_Y Y_2}$ , and we set  $g_3 : Y_3 \rightarrow Y$  to be the composite

$$g_3 := g_1 \circ \widetilde{g}_2 \circ \pi = g_2 \circ \widetilde{g}_1 \circ \pi.$$

Which means that  $g_1 \leq g_3$  and  $g_2 \leq g_3$ .  $\square$

Therefore we can see that the collection  $\mathcal{RES}_Y$  of resolution of singularities becomes a projective system with the index set being the directed set  $\mathcal{RES}_Y$  itself.

**Definition (7.2).** For a possibly singular variety  $Y$ , the projective limit of the projective system  $\mathcal{RES}_Y$  of resolutions of singularities of  $Y$  is called *the proresolution of  $Y$*  (a sort of “maximal” resolution of  $X$ ) and denoted by

$$\pi^{\text{pro}} : \widetilde{Y}^{\text{pro}} \rightarrow Y.$$

This proresolution is motivated by our previous work [Y4]. Our naïve question is a relationship between the Nash arc space and the proresolution:

**Question (7.3).** *Let  $X$  be a possibly singular variety. Then does there exist a canonical promorphism  $\nu : \mathcal{L}(X) \rightarrow \widetilde{X}^{\text{pro}}$  as a  $X$ -provariety ?, i.e., such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{L}(X) & \xrightarrow{\nu} & \widetilde{X}^{\text{pro}} \\ & \searrow & \swarrow \\ & X. & \end{array}$$

One could ask if there exists a canonical morphism from  $\widetilde{X}^{\text{pro}}$  to  $\mathcal{L}(X)$  over  $X$ , but it would be reasonable to consider the above question, because over the nonsingular part of  $X$  the proresolution  $\widetilde{X}^{\text{pro}}$  is isomorphic to the nonsingular part of  $X$ , whereas the Nash arc space  $\mathcal{L}(X)$  is a fiber bundle over the nonsingular part  $X_{\text{smooth}}$  of  $X$  with fiber being the proalgebraic variety  $\mathbb{C}^{\mathbb{N}}$ , to be more precise,  $X_{\text{smooth}} \widetilde{\times} (\mathbb{C}^{\dim X})^{\mathbb{N}}$ , using the notation given in §5. We hope to return to this question.

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF SCIENCE, UNIVERSITY OF KAGOSHIMA, 21-35 KORIMOTO 1-CHOME, KAGOSHIMA 890-0065, JAPAN

*E-mail address:* yokura@sci.kagoshima-u.ac.jp